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ISSUE No. 6

PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

SOLID GEOMETRY

Points in Space
Polyhedral Angles
Areas and Volumes of Solids
Simple Engineering Drawing
Coördinate Planes in Mechanical Drawing

IRRATIONAL AND IMAGINARY NUMBERS

ALSO

Mathematical Tables and Formulas
Glossary of Mathematical Terms
Self-Tests and Geometry Problems

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35¢

EDITOR: REGINALD STEVENS KIMBALL ED.D.

Issue No. 6 PRACTICAL MATHEMATICS Volume No. 1

ISSUE 6 **Practical Mathematics** VOLUME 1

REGINALD STEVENS KIMBALL, Editor

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CHATS WITH THE EDITOR

WITH this issue on solid geometry, PRACTICAL MATHEMATICS passes to a phase of mathematics which many high school students never meet. For some reason, the minimum requirements in the study of mathematics have been set at the point where the student gets a glimpse of plane geometry and a smattering of advanced algebra. Many colleges require only this amount of mathematics as a prerequisite to entrance.

Students of mathematics in vocational schools are a bit more fortunate. Because of the many practical uses to which solid geometry may be put, they are given some work in this subject, though usually they do not get the complete view of solid geometry which is desirable for a full understanding of the work they will be required to do with it.

Following the plan already used in presenting plane geometry, PRACTICAL MATHEMATICS is avoiding a plethora of formal proofs in the field of solid geometry, confining the treatment to a consideration of formulae which will be needed in actual-life situations. For the author of this section, we have selected Dr. Edward Kasner, co-author of the best seller, *Mathematics and Imagination.*

In solid geometry, we deal first with box-like objects built up on rectangular bases, then proceed to bases of other polygons, such as the triangle and the pentagon, and then consider the special cases in which the sides of the boxes are not perpendicular to the base, getting pyramids or parts of pyramids and cones, or parts of cones. Since containers for many goods are of these shapes, the reader will have no difficulty in seeing the practical applications of this material.

A knowledge of the sphere and its properties will be found especially helpful when we come to the later issues in which we deal with such subjects as navigation and aviation. For this reason, we shall find it desirable in rounding out our knowledge of solid geometry to get a good understanding of the mathematical principles involved in the measurement of the sphere.

Throughout his discussion, Dr. Kasner calls to your attention the relationships between the facts you have learned in plane geometry and the new facts which you are learning in solid geometry. You should have little difficulty in extending your knowledge from the plane to the solid.

Many students "bog down" in solid geometry because they are not able to "see through" the diagrams. When we try to represent on a two-dimensional plane the appearance of an object which has three dimensions, we are apt to forget some of the hidden, inner lines. In our diagrams, these lines appear as dotted lines. This device is used to help your eye convey to your brain the fact that you are looking *inside* the figure.

You may find it helpful to construct for yourself some of the diagrams represented in the book. For this purpose, lengths of fairly stiff wire (in the days when pipe-cleaners were plentiful, I used to suggest that my students stock up on them) may be used to represent lines. To show a plane, fasten a sheet of paper or thin bristol board to those wires which lie in the same plane. By placing a flash-

light inside the figure, you may bring into view the construction lines on each plane of the figure.

If you prefer to make more elaborate models, you may use slender dowel-sticks to represent the lines and sheets of glass or of cellophane to represent the planes. The dowel-sticks have one advantage over the lengths of wire: they are less flexible and hence give you more rigidity when dealing with straight lines. For the same reason, however, they are less adaptable to work with curved lines.

For understanding the sphere, you will find that any rubber ball is of assistance. By piercing the ball with hatpins representing lines, you may judge something of the relationship of lines within the sphere. Chalk-lines drawn on the surface will give you the representation of great circles or of other arcs.

Let me suggest that you retain for future reference any models which you construct at this time. We shall have occasion to refer to them several times before the course is over.

Several times in the course of our work in advanced algebra, we have come upon irrational and imaginary numbers, but have deferred consideration of their special characteristics. It is now time that we make good this gap. To that end, we are including in the present issue a consideration of both positive and negative surds, showing the reader how to employ them in the solution of problems which otherwise would remain unsolved. Don't pass up this article just because it doesn't seem to have an immediate meaning for you. It is one of the keys which serve to unlock the mysteries of mathematics.

With the background of plane and solid geometry well in hand, we shall be ready in Issue Number Seven to discuss trigonometry, the measure of angles. Dr. Agnew will assist you in this subject to understand why trigonometry has come to be regarded as a special field of great importance in connection with many war-time problems in mathematics. No longer need trigonometry be considered an abstruse branch of mathematics, suitable only for advanced students. In fact, it has been customary, for many years past, to include a brief explanation of trigonometry in many of the elementary courses in algebra and plane geometry in the senior high schools, and, to some extent, in the junior high schools. In our treatment of the subject, in PRACTICAL MATHEMATICS, we shall confine ourselves to those phases for which you will find immediate use in connection with your chosen fields of endeavor.

The work in trigonometry will prepare you for an introduction to the calculus, which will be presented in Issue Number Eight by Mr. Harvey and Dr. Wiener. Following that, we shall go on, in Issue Number Nine, to a consideration of differential equations, by Dr. Menger, and a special article on mensuration, prepared by Mr. Baker, the assistant director of the War Industries Training School at Stevens Institute of Technology.

That will round out our treatment of theoretical mathematics. In the issues in which these subjects are considered, we shall follow the practice of presenting practical problems drawn from the fields of industry and military and naval science, in order that you may have opportunity immediately to put your new-gained knowledge to use.

Beginning with Issue Number Ten, we shall devote four issues to various branches of applied mathematics. These are the issues for which some of you have been waiting eagerly. In the course of Issues Ten through Thirteen, we shall give you an opportunity to review much of the mathematics you have been learning in the

previous issues and, at the same time, we shall be giving you a wider knowledge of subjects for which you will have definite use in the days to come.

Issue Number Ten will contain an article on structural engineering, by Dr. Sollenberger, and an article on the mathematics of the machine shop, by Mr. Benedict. In Issue Number Eleven, we shall consider the mathematics of heat and chemistry, under the guidance of Dr. Hynes and Dr. Pickering. In Issue Number Twelve, we turn to electricity, which, with the allied field of radio, has an important bearing in present-day affairs. Here Dr. Hefner will present the main treatment. Coming to Issue Number Thirteen, we shall have articles on navigation and aviation, written by Dr. Michael, and gunnery and ballistics, by Mr. Littauer.

The fourteenth issue will complete the course. Dr. Curtiss, the President of the Mathematical Association of America, will present an article showing general applications of mathematics to many fields not touched upon in previous issues and will direct your thinking toward a review of all that has preceded. In the fourteenth issue, also, we shall print a comprehensive index to the entire series, so that you may have at your finger-tips a guide through the wealth of material which has come to you in the 896 pages of the course.

Don't overlook the tables which we have been presenting in connection with each issue. Rather than to wait for the last number and insert the tables in a body, we have been printing a few tables in each issue, giving you each time those tables for which you will have immediate use in connection with the subjects already presented. We have placed the tables in the back part of each issue, so that you may find them readily if you have occasion to refer to them. Such tables as those dealing with factors, roots, and logarithms will be needed so frequently that you should memorize their location and be able to turn to them without loss of time.

Once again, let me remind you that the way to master mathematics is to work with it. Constant practice on the principles which are presented will be necessary if you are to get a complete understanding of the subject-matter. The group of exercises following each new section is intended to give you an immediate opportunity to test your understanding of the new subject. Careful attention to the solution of all the exercises in the group will be good insurance against your getting befuddled as the author proceeds with the next section. Stop long enough to ponder on what you have read. You will find that your haste will be the more rapid because of these pauses.

Perhaps you will be as interested as I have been in learning just what sort of students make up our PRACTICAL MATHEMATICS family. From the letters which have been coming in, I can assure you that there are no age-limits in this group. Men and women well past 60 years of age are included among our subscribers. Some of them had studied mathematics years ago, but had not had much occasion to use the subject in the intervening years. Now, because they are called upon to utilize mathematics in the work they are doing in defense plants, they desire to brush off the cobwebs which have gathered during the years. Some of the older men and women among the subscribers are attempting to extend their knowledge of mathematics past the point to which they carried their study of the subject in their school days. Many of these are enthusiastic about the practical "slant" which they are gaining through this course.

From the training camps and drill fields in various parts of the country,

men in training are writing to me of the ways in which PRACTICAL MATHEMATICS has already helped them "get ahead" in winning a desired promotion. These young men—some of them fresh out of high school—are finding that the odd problems are giving them a means of whiling away off hours in camp. They ponder over the possible solutions to some of the puzzles; then, in their eagerness, they write in for help in settling arguments which have come up among their buddies. Since we are printing the explanations of the solutions in the following issue, I suggest that you restrain your curiosity for just a few days; you have only 10 days to wait until the next issue comes along with the answer to the puzzle. Meanwhile, if at first you don't succeed, try, try again.

Are you keeping up your practice on the slide rule? As the days go by and the subject-matter of PRACTICAL MATHEMATICS introduces more and more opportunities to use the slide rule in solving the numerical part of the example, you will find that facility in its use will free you to devote more time to reasoning out the steps that are to be taken in solving a problem. Both the slide rule and the log tables should have become matters of habit by this time. Brush up on them, as we shall be suggesting to you that you extend your use of these instruments when you turn to trigonometry in the next issue.

R. S. K.

ABOUT OUR AUTHORS

EDWARD KASNER, Adrain Professor of Mathematics at Columbia University, is the co-author of a "best seller", *Mathematics and Imagination*, which has been widely accepted as one of the best and most readable popular treatments of mathematics. After receiving the degree of Bachelor of Science from the College of the City of New York in 1896, Mr. Kasner went to Columbia University, where he earned the degree of Master of Arts in 1897 and the degree of Doctor of Philosophy in 1899. The following year he spent in Europe, studying at the University of Göttingen.

Returning to Columbia, he served as a tutor in mathematics from 1900 to 1905, as instructor from 1905 to 1906, as Adjunct Professor from 1906 to 1910, attaining full professorial rank in the latter year. In 1937, he was installed in the chair of the Adrain Professorship. He lectures also at the New School for Social Research and at St. John's College, Annapolis.

He is a member of Phi Beta Kappa, the National Academy of Sciences, the American Mathematical Society, the American Association for the Advancement of Science, the Circolo Matematico di Palermo, and the Société Mathématique of France. He has held office in several of these organizations and has been a member of the mathematics committee of the National Research Council.

Dr. Kasner's writings have been prolific, his contributions of articles on the Einstein theory, conformal geometry, and polygenic functions running to several hundred. He is the editor of the *Transactions* of the American Mathematical Society and has appeared on many lecture platforms. He was a delegate to the International Mathematics Congresses held at Bologna, Rome, and Zurich. In 1936, he was awarded the Townsend Harris medal in recognition of his conspicuous contributions to learning.

· ELEMENTS OF SOLID GEOMETRY ·

By Edward Kasner, Ph.D.

IN SOLID geometry, we study the properties and the relations of objects in three dimensions. Space is three dimensional. Everything around us can be measured in terms of *width*, *length*, and *height*. Thus, the measurements of solid objects can be obtained when we can reduce them to the above three.

FUNDAMENTAL ELEMENTS

The fundamental elements that constitute the outlines (or boundaries) of solid objects are: *points*, *lines*, *planes*, and *surfaces*. By means of these four elements, when they are properly combined, we obtain a solid object, such as a cube, a pyramid, or a cone.

When we study the various properties of the elements that make up solid objects as well as their various relationships, we shall make use of the simple properties of lines and angles that were obtained in plane geometry. It will help the reader to refer to Issue Number Five of PRACTICAL MATHEMATICS whenever he is in doubt or he wishes to recall them.

In plane geometry, the reader obtained a fairly correct notion of lines, straight or curved, and angles. When speaking of a *plane*, we mean a perfectly flat surface, such as the surface of a table. A plane has no thickness, but it has length and width, and it can be extended indefinitely in all directions. Another important property of a plane is that all the points contained in a straight line drawn in it lie within the plane. Furthermore, since a line is determined by two points, if we take any two points in a plane, they may be joined by a straight line.

Determining planes

In plane geometry, we learned that, through any one point, an infinite number of straight lines may be drawn (one point does not determine, or fix, a straight line). In solid geometry, we have a similar

situation, but here the straight line does the work of the point; that is, one line does not determine, or fix, a plane.

To understand this, take a rectangular piece of cardboard, holding it with two fingers at points on the opposite edges. These two points may be thought of as determining a straight line. Now rotate the cardboard. As it rotates, it assumes various positions, an infinity of them. In each position, we have a plane. Thus, we find that, through one straight line, an infinity of planes may be drawn. (Fig. 1.)

In order to fix (determine) a plane, we take the revolving plane and stop it at some point (one that would obstruct the plane in its revolution). Thus, a straight line and a point outside the line fix (determine) a plane.

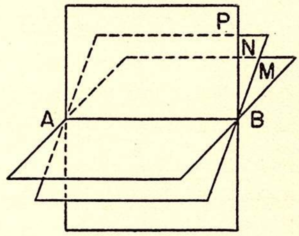

Fig. 1

From the fact just obtained, the following can be derived immediately. If we have a straight line and a point outside of it, we can draw a straight line through that point which will either intersect that first line or will be parallel to it. This leads us to the conclusion that a plane is determined:

a by two intersecting straight lines, **I**

b by two parallel straight lines. **II**

Also, if we have three points which determine two intersecting straight lines, —that is, three points not collinear—, these three points also determine a plane. **III**

These results find many practical applications.

When two lines are in such positions that a plane cannot be passed through them, they are known as *skew lines*.

TEST YOUR KNOWLEDGE OF PLANES WITH THESE QUESTIONS

1 When a sidewalk is cemented or a plasterer puts plaster on a wall, the wet surface is smoothed with a board. Why is this done?

2 Why are telescopes, cameras, and surveying instruments mounted on tripods?

3 Why are some chairs and tables wobbly?

4 Hold two pencils in various positions and show that under certain conditions a plane cannot be passed through them.

5 Do four points always lie in a plane?

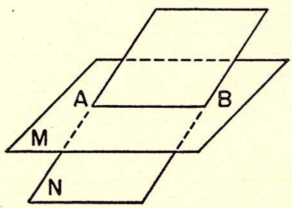

Fig. 2

Intersections of planes

When two planes intersect, they have some points in common. Two intersecting planes have at least two points in common. Since through any two points in a plane a straight line can be drawn, two planes intersect in a straight line. **IV**

The straight line, AB (Fig. 2), lies in the two intersecting planes, M and N; that is why it is common to the two of them.

Three intersecting planes do not always intersect in a straight line; they sometimes intersect at a point. **V**

You can note this by observing a corner of a room (Fig. 3), where *ABCD* is the floor, *ABFE* the left wall, and *BCGF* the rear wall. The three planes intersect at the point, *B;* planes *L* and *R* intersect at the line, *FB;* planes *P* and *R* intersect at the line, *BC;* while planes *P* and *L* intersect in the line, *AB*.

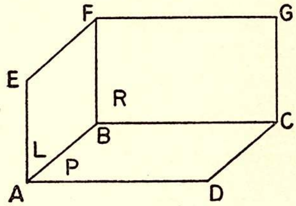

Fig. 3

When three planes do not intersect in a point, they intersect in straight lines by pairs or they intersect in one straight line.

In Fig. 4, planes *M* and *N* are shown intersecting in the line, *AB;* planes *M* and *Q* intersect in the line, *CD*, and planes *N* and *Q* intersect in the line, *EF*. Fig. 1, on the other hand, shows the planes, *M*, *N*, and *P* all intersecting in the line, *AB*.

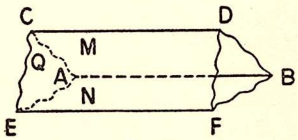

Fig. 4

When three planes intersect in straight lines by pairs, the straight lines are parallel to each other. **VI**

In Fig. 5, plane *M* intersects *Q* in the line, *AB*, and *N* intersects *Q* in the line, *CD*. Then *AB* and *CD* are parallel.

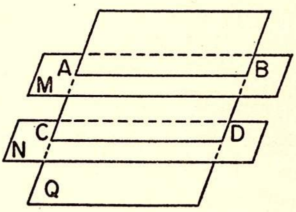

Fig. 5

If this were not so, the straight lines, would intersect in a point, and then the planes would have this point as common to all of them (Fig. 3).

TEST YOUR KNOWLEDGE OF INTERSECTING PLANES WITH THESE EXERCISES

6 What is the locus of all points common to two intersecting planes?

7 A carpenter who wishes to cut a board lengthwise fastens a taut, chalked string to both ends of the board. Then he pulls the string slightly, and lets it go. When the string hits the surface of the board, a straight line is marked on the surface. Why?

8 How many planes can be passed through a line in a given plane?

9 Any two surfaces intersect in a straight line. Are these two surfaces necessarily planes?

10 Can a triangle be placed in a plane? Why?

11 When one saws through a wooden board, why is the edge a straight line?

PERPENDICULARS

A straight line that is perpendicular to a plane is perpendicular to every straight line in that plane that is drawn through the point where the perpendicular cuts the plane. **VII**

We may also say that, when a straight line is perpendicular to a plane, the plane is perpendicular to the straight line. **VIII**

O in Fig. 6 is the foot of the perpendicular, *AB*.

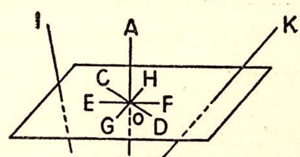

Fig. 6

A straight line that is not perpendicular to a plane is oblique to it or is parallel to it.

IJ and *KL* in Fig. 6 are examples of oblique lines.

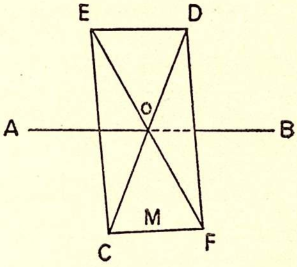

Fig. 7

If we have a straight line, a plane can be drawn perpendicular to it. This can be accomplished as follows:

At some point, *O* (Fig. 7), on the straight line, *AB*, two perpendiculars, *CD* and *EF*, are drawn to the straight line. Through these two perpendiculars, a plane, *M*, is drawn. This plane will be perpendicular to the straight line, *AB*.

Note that only one perpendicular plane can be drawn at a given point on a straight line. IX

In Fig. 8, we assume that two such planes, *M* and *N*, can be drawn perpendicular to the line, *AB*. We then draw through the foot, *O*, a straight line in each plane. Through these two straight lines, *OC* and *OD*, and the perpendicular, *AB*, we draw another plane, *P*. (Now we revert to plane geometry.)

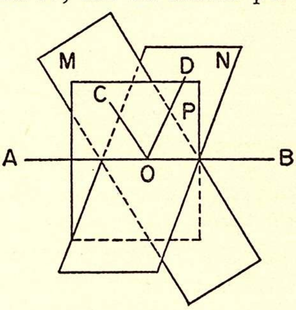

Fig. 8

We then have an impossible situation that two perpendiculars are drawn to a straight line at the same point, the straight line and the perpendiculars all lying in the same plane.

From the above, we immediately conclude that all the perpendiculars to a straight line at a given point lie in the same plane. X

TEST YOUR KNOWLEDGE OF PERPENDICULARS WITH THESE EXERCISES

12 Does the spoke of a wheel which is perpendicular to the axle describe a plane when the wheel turns?

13 A carpenter often uses a level such as that shown in Fig. 9. Explain how this level works.

14 Can we have two different perpendiculars drawn from a point outside a plane to the same plane? Explain.

15 If we have two different perpendiculars to the same plane, can they pierce the plane in the same point?

16 Do two different perpendiculars to the same plane lie in a plane? (*Hint:* Pass a plane through these perpendiculars and revert to plane geometry.)

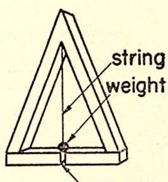

Fig. 9

Lengths of perpendiculars to a plane

The shortest distance to a plane from a point outside is measured along a perpendicular to that plane from the given point. XI

This is shown in Fig. 10, where we have the plane, *P*, with the line, *AO*, drawn perpendicular.

Through the perpendicular, *AO*, we draw a plane, *Q*. This plane is perpendicular to the original plane. (Again we revert to plane geometry.)

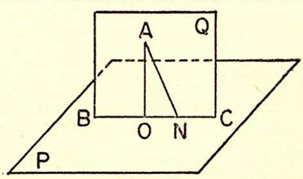

Fig. 10

In the plane that was drawn, the shortest distance from the point, *A*, to the straight line, *BC* (the intersection of the two planes), is measured along the perpendicular, *AO*.

Any other line drawn from a point outside a plane to the plane is oblique and its length is greater than that of the perpendicular (Fig. 11). **XII**

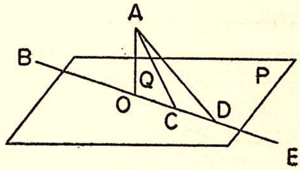

Fig. 11

In Fig. 11, *AC* and *AD* are examples of such oblique lines.

If, from a point outside a plane, a perpendicular and an oblique line are drawn, the angle that the oblique line makes with the plane is smaller than a right angle. **XIII**

Also, with the foot of the perpendicular as a center and the line joining the point where the oblique line cuts the plane and the foot of the perpendicular as a radius, a circle can be drawn. This arrangement is very important in sea navigation. If the angle that the oblique line makes with the plane is known, this angle may be utilized when dangerous points, such as shoals or rocks, need to be avoided.

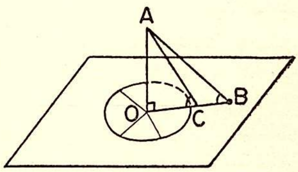

Fig. 12

The captain of a ship observes a point (such as *A* in Fig. 12), takes note of the angle, *ACO* (which is usually known to him), and guides his ship so that he can observe a smaller angle, *ABO*. The principle that is involved in this case is that, the farther away his ship is from the center of the circle, the smaller is the angle that the oblique line makes with the plane.

TEST YOUR KNOWLEDGE OF OBLIQUE LINES WITH THESE EXERCISES

17 From a point outside a plane, two equal oblique lines are drawn. What can be said of the angles that these oblique lines make with the plane?

18 If two lines are perpendicular to a plane, are they parallel? (*Hint:* Pass a plane through these two perpendiculars and revert to plane geometry.)

19 If two planes are perpendicular to the same straight line, are they parallel? (*Hint:* Pass a plane through the straight line so that the straight line lies in it and revert to plane geometry.)

20 How is the distance between two parallel planes measured?

LOCATING POINTS IN SPACE

In coördinate geometry of the plane, a point is located when it can be referred to two coördinate perpendicular axes. The location of the point is given by two numbers, or coördinates of the point. These two numbers represent the respective distances of the point from the two coördinate axes.

If the location of a point in space is desired, two distances are not sufficient for fixing this location exactly. This can be observed by considering the location of some point in a room. In order to locate this point exactly, we must know its distance from the floor, and from the front and side walls. When we talk of the distances, we mean the shortest distances from planes. These,

as we now know, are measured along perpendiculars drawn from the point to the various planes. Thus, in order to locate a point in space, we must have three planes which are, by agreement, perpendicular to one another, as the two walls and the floor (or the ceiling) of a room. This is shown in Fig. 13.

These planes perpendicular to one another are known as the *coördinate planes*. They intersect by pairs in straight lines. There are three of them, and they are known as the *coördinate axes*. These three coördinate axes are also mutually perpendicular.

Following the same method of agreeing on directions as we used in plane coördinates (Issue Number Five, page 274), we designate the directions in three coördinates as follows (Fig. 13):

	Positive	Negative
Axis $X'OX$	to the right	to the left
Axis YOY'	toward you	away from you
Axis ZOZ'	upward	downward

All these are with reference to the origin, O, whose coördinates are (0,0,0).

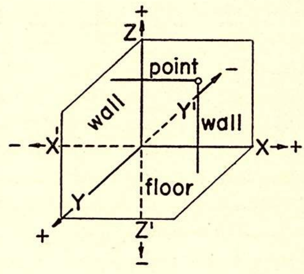

Fig. 13

The coördinates are recorded always in the order, (x, y, z). The first is the coördinate, x, which is measured along the axis, $X'OX$, giving the distance from the plane, M. The second is the coördinate, y, which is measured along the axis, YOY', giving the distance from the plane, N. The third is the coördinate, z, which is measured along the axis, ZOZ', giving the distance from the plane, P.

TEST YOUR KNOWLEDGE OF COÖRDINATES WITH THESE EXERCISES

Draw three coördinate axes and locate the following points:

21 (2, 0, 0) **22** (−2, 0, 0) **23** (0, 3, 0) **24** (0, −3, 0) **25** (0, 0, 4)

Locate the following points:

26 (2, 3, 0)	**27** (−2, 3, 0)	**28** (2, −3, 0)	**29** (−2, −3, 0)
30 (2, 0, 3)	**31** (−2, 0, 3)	**32** (2, 0, −3)	**33** (−2, 0, −3)
34 (0, 2, 3)	**35** (0, −2, 3)	**36** (0, 2, −3)	**37** (0, −2, −3)
38 (2, 3, 4)	**39** (−2, 3, 4)	**40** (2, −3, 4)	**41** (2, 3, −4)
42 (−2, −3, 4)	**43** (−2, 3, −4)	**44** (2, −3, −4)	**45** (−2, −3, −4)

Distance between two points

In plane coördinate geometry, we obtained a formula for the distance between two points whose coördinates are (x_1, y_1) and (x_2, y_2). This formula may be expressed as

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

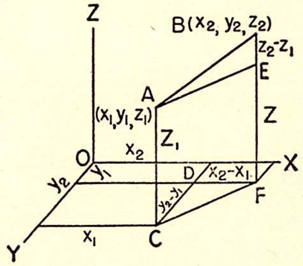

Fig. 14

In space, we have two points whose coördinates are $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$. The formula for the distance, AB, may be obtained by reverting to plane coördinate geometry as follows; this is shown in Fig. 14.

$$CE=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$
$$AB^2=CE^2+AE^2.$$

But $\qquad AE=z_2-z_1.$

Then $\qquad AB=d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$

The distance between the two points, $A(1, 2, 3)$ and $B(3, 4, 5)$, is

$$AB=\sqrt{(3-1)^2+(4-2)^2+(5-3)^2}$$
$$=\sqrt{2^2+2^2+2^2}=\sqrt{4+4+4}=\sqrt{12}=3.4641.$$

TEST YOUR KNOWLEDGE OF POINTS IN SPACE WITH THESE PROBLEMS

46 Find the distance between the two points, $A(0, 3, 5)$ and $B(4, 0, 5)$.
47 Find the distance between the two points, $A(2, 4, 6)$ and $B(5, 7, 9)$.
48 The distance between two points is 10. The coördinates of the points are $A(1, 2, 3)$ and $B(x, 7, 8)$. What is the X-coördinate of B?
49 What is the distance of the point, $A(x, y, z)$, from the origin?

SIMPLE ENGINEERING DRAWING

By means of the method of space coördinates, we can learn the methods of engineering drawing.

The projection of a point on a plane is the foot of the perpendicular drawn from that point to the plane (Fig. 6).

The projection, CD, of a straight line-segment, AB, on a plane, M, is obtained by drawing perpendiculars, AC and BD, from the end points of the line-segment, AB, to the plane, M, and joining the feet of these perpendiculars, CD (Fig. 15). The projection is a line-segment.

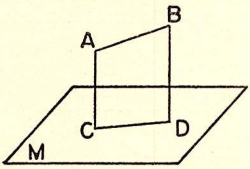

Fig. 15

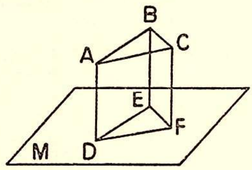

Fig. 16

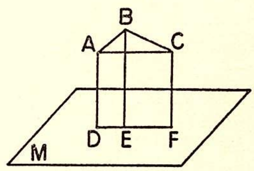

Fig. 17

The projection of a triangle, ABC, on a plane, M, is obtained by drawing three perpendiculars, AD, BE, and CF, from the vertices of the triangle to the plane and joining the three feet of the perpendiculars.

The projection is a triangle, DEF, if the plane of the projected triangle is oblique to the plane. (Fig. 16.)

The projection is a line-segment, DF, if the plane of the projected triangle is perpendicular to the plane, as in Fig. 17.

TEST YOUR KNOWLEDGE OF PROJECTIONS OF LINES WITH THESE EXERCISES

50 Can the projection of a straight line ever be a point?
51 When is the length of the projection of a line-segment equal to the length of the projected line-segment?
52 Under what conditions will the area of a projected triangle be equal to the area of its projection?

53 Is it possible to project a triangle so that the area of the projection is not equal to the area of the projected triangle, but one side of the triangle remains equal to the corresponding side of its projection? (*Hint:* Cut out a triangle from cardboard and hold it under a strong light in various positions over a table, observing the shadow cast by the triangle onto the table.)

54 If the areas of a projected triangle and of its projection are not equal, are the corresponding angles of the two triangles equal?

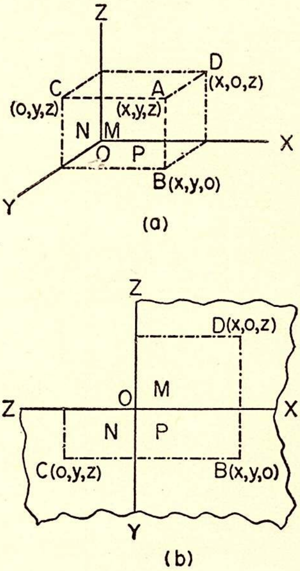

Fig. 18

USING COÖRDINATE PLANES IN MECHANICAL DRAWING

By this time, the reader will have noticed that a point in space can be projected onto the three coördinate planes. Furthermore, if the three coördinates of a point are given, the locations of its projections onto the three coördinate planes are definitely fixed.

This can be seen in Fig. 18. Fig. 18b shows the three coördinate planes flattened out.

Note that, in Fig. 18a, perpendiculars, *AB*, *AC*, and *AD* are drawn from the point, *A*, to the respective coördinate planes.

The feet of these perpendiculars, *B*, *C*, and *D*, are the respective projections of the point.

Since only one perpendicular can be drawn from a point outside the plane to a plane, there can be only one foot of this perpendicular, and its location is fixed as explained below.

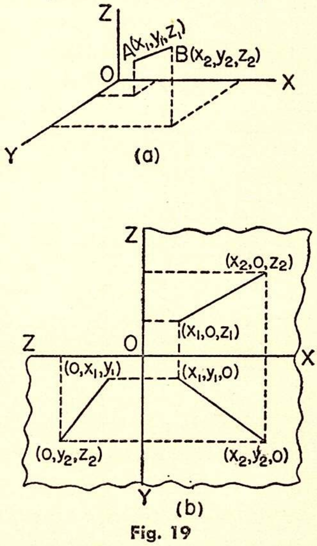

Fig. 19

The coördinates of the three projections of a point onto the three coördinate planes are the respective coördinates of the points taken by pairs. However, they are not taken at random. They are selected so that they correspond to the plane onto which the point is projected.

If the point is projected onto the XOY-plane, the coördinates of the projection are x and y. The Z-coördinate is 0 (zero). If the point is projected onto the XOZ-plane, the coördinates of the projection are x and z. The Y-coördinate is 0 (zero). If the point is projected onto the YOZ-plane, the coördinates of the projection are y and z. The X-coördinate is 0 (zero). This is shown in Fig. 18b.

The projections of a line-segment onto the three coördinate planes are obtained by projecting the end-points of the line-segment (*AB*, their coöordinates

are given), onto the three coördinate planes as shown in Fig. 19 and then joining the pairs of projected points in the three planes.

TEST YOUR KNOWLEDGE OF PROJECTIONS OF POINTS WITH THESE EXERCISES

55 Obtain the projections of the point (3, 4, 5).
56 Obtain the projections of the point (0, 2, 4).
57 What can be said of the projections of a point one of whose coördinates is zero?
58 Obtain the projections of AB, when A is (1, 3, 5) and B is (3, 6, 7).
59 Can one of the projections of a straight line-segment be a point?
60 Can two of the projections of a straight line-segment be points?
61 Can all three projections of a straight line-segment be points?

Lengths of the projections of a line-segment

Since the coördinates of the end-points of each projection of a line-segment are known (see above), it is possible to compute the length of each projection.

For example, the line-segment, AB, $A(3, 4, 5)$ and $B(6, 7, 8)$, is projected onto the three coördinate planes. The coördinates of the end-points of the three projections are:

On the XOY plane, (3, 4, 0) and (6, 7, 0)
On the XOZ plane, (3, 0, 5) and (6, 0, 8)
On the YOZ plane, (0, 4, 5) and (0, 7, 8)

The lengths of the projections are:

On the XOY plane, $\sqrt{(6-3)^2+(7-4)^2}=\sqrt{3^2+3^2}=\sqrt{9+9}=\sqrt{18}=4.24$
On the XOZ plane, $\sqrt{(6-3)^2+(8-5)^2}=\sqrt{3^2+3^2}=\sqrt{9+9}=\sqrt{18}=4.24$
On the YOZ plane, $\sqrt{(7-4)^2+(8-5)^2}=\sqrt{3^2+3^2}=\sqrt{9+9}=\sqrt{18}=4.24.$

In order that one of the projections of a straight line-segment be a point, it is necessary that the length of the projection be equal to zero. This can be attained when the end-points of the line-segment have two pairs of coördinates equal (the x of one is equal to the x of the other, and the y of one equal to the y of the other, or the x's are equal and the z's are equal, or the y's are equal and the z's are equal).

For example, the straight line-segment whose end-points are $A(5, 7, 8)$ and $B(4, 7, 8)$ has a point as its projection onto the YOZ plane because the y's and z's are equal, and

$$\sqrt{(7-7)^2+(8-8)^2}=\sqrt{0+0}=0.$$

A triangle can be projected onto the three coördinate planes in the same manner as a straight line-segment. Actually, the projection of a triangle is reduced to the projection of its three vertices. After this is accomplished, the projections onto the three planes are three points,

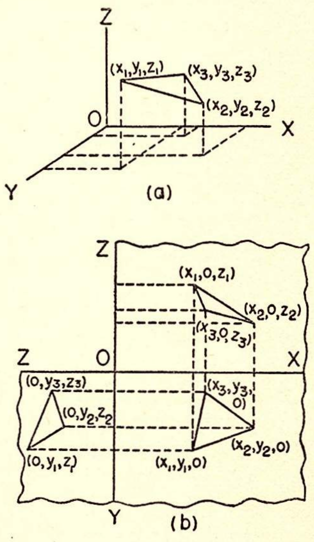

Fig. 20

and these are joined by straight lines. Thus, in the general case, we obtain three triangles which are the respective projections.

This is shown in Fig. 20.

TEST YOUR ABILITY TO MEASURE PROJECTIONS WITH THESE EXERCISES

62 What are the lengths of the projections of the straight line-segment, AB, $A(2, 5, 3)$ and $B(1, 7, 6)$?

63 If the projection of a straight line-segment onto one of the planes is a point, what is the relation of the straight line-segment to one of the other two planes?

64 Obtain the projections onto the three coördinate planes of the triangle, ABC, $A(2, 7, 5)$, $B(4, 1, 9)$, and $C(3, 6, 8)$.

65 Can one of the three projections of a triangle be a straight line? If so, when?

66 Can two of the projections of a triangle be straight lines? If so, when?

67 Can one of the projections of a triangle be a point? If so, when?

68 Compute the lengths of the sides of the three triangles obtained in Problem 64.

ANGLES FORMED BY PLANES

When two planes intersect, they form an angle. This angle is different from the angles encountered in plane geometry (as is shown in Fig. 21a), because its sides are planes, M and N (called the *faces* of the angle), and, in place of a point as a vertex, we have a straight line, AB (called the *edge* of the angle). Such an angle is known as the *dihedral angle* (an angle with two faces). A most common illustration of such an angle is obtained when opening a book (if we think of the pages as representing planes).

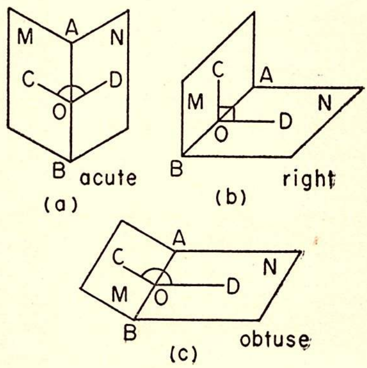

Fig. 21

Dihedral angles

A dihedral angle is measured in terms of plane angles. This measure is obtained by forming a plane angle as follows:

At some point on the straight line which is the intersection of the two planes, such as O, perpendiculars (CO and DO) are drawn to the edge so that they lie in the two faces, M and N. These two perpendiculars form an angle, COD, which is used for measuring the dihedral angle. Thus, if the angle so formed measures 35°, we say that the dihedral angle to which it corresponds also measures 35°.

It should be noted that two very simple facts may be derived from the relation between the dihedral angle and its corresponding plane angle. These are:

If two dihedral angles are equal, their plane angles are equal. **XIV**

Of two unequal dihedral angles, the greater will have a greater plane angle. **XV**

Sizes of dihedral angles are classified in the same way as the sizes of plane angles (see page 259, Issue Number Five).

A Dihedral Angle is Called	If its Plane Angle Measures
Acute	Less than 90°
Right	90°
Obtuse	More than 90°

You will remember, from the discussion in plane geometry, that the whole angle around a point measures 360°. In solid geometry, a straight line plays the same part which a point plays in plane geometry. Thus, around a straight line, we have a dihedral angle of 360°.

In plane geometry, two intersecting straight lines form a figure, *X*, with four angles, and the non-adjacent angles (usually called vertical) are equal. Again, two intersecting planes, *M* and *N*, form a solid figure, *X* (Fig. 22), and the non-adjacent dihedral angles are equal. **XVI**

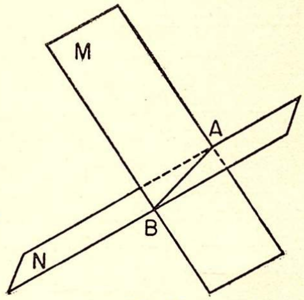

Fig. 22

TEST YOUR KNOWLEDGE OF DIHEDRAL ANGLES

69 If two planes are perpendicular, what kind of dihedral angles do they form?

70 Draw a right dihedral angle and then draw its plane angle. What is the relation of the edge of the dihedral angle to a plane passed through this plane angle?

71 Draw a dihedral angle and then pass a plane parallel to the edge of this angle. What can be said of the lines formed by these intersections?

72 What is the size of the dihedral angle about a straight line drawn in a plane?

Some loci in space

An oblique line makes an angle with a plane to which it is drawn.

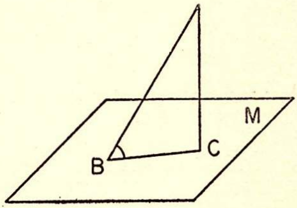

Fig. 23

In order that we may have a clear notion of the measure of this angle, let us agree that the measure of this angle is obtained by drawing, from some point on the oblique line, *AB* (Fig. 23), a perpendicular, *AC*, to the plane, *M*. Then, from the point, *B*, where the oblique line, *AB*, pierces the plane, *M*, a line, *BC*, is drawn to the foot of the perpendicular. The measure of the angle that the oblique line, *AB*, makes with the plane, *M*, is obtained by measuring the angle, *ABC*, formed by the oblique line, *AB*, and the line, *BC*, drawn to the foot of the perpendicular, *AC*. This is the angle, *ABC*.

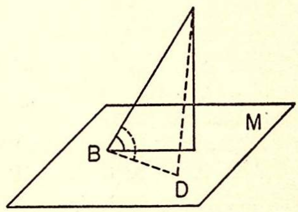

Fig. 24

It can be observed that this angle, *ABC*, is smaller than any other angle that can be considered by taking the oblique line, *AB*, and the plane, *M* (Fig. 24). **XVII**

In order to convince ourselves of this fact, we need to refer to plane

geometry. Of all possible triangles with AB as one side and the other sides drawn as shown in Fig. 24, angle ABC is the smallest.

Two parallel oblique lines drawn to the same plane make the same angle with the plane. **XVIII**

This a generalization of the fact that if one of two parallel lines is perpendicular to a plane, then the other is also. **XIX**

In plane geometry, an angle bisector (a line dividing an angle into two equal parts) is such that any point on it is equally distant from the two sides of the angle. **XX**

In solid geometry, the same holds true. This can be seen from the following (Fig. 25).

In the dihedral angle, XOY, we drew two plane angles, AOB and COD, that measure it. These two plane angles are equal.

Then we drew the angle bisectors, OP and OQ, of these two plane angles.

Through these two plane angles, we passed a plane, L.

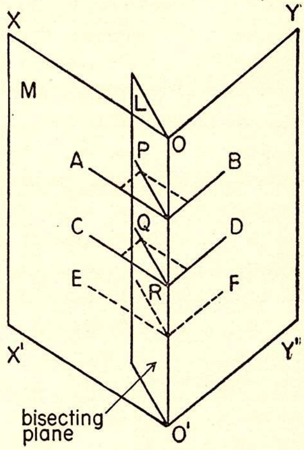

Fig. 25

Now, the angle bisectors of the plane angle have points (all of them) that are equally distant from the sides of the angles, and so from the faces of the dihedral angle.

If we draw any other plane angle, as EOF, in this figure and its angle bisector, R, the same situation obtains.

It can be then observed that the generalization of the fact from plane geometry is applicable to solid geometry.

If we take a point on a line perpendicular to a plane, then this point is always equally distant from the points on a circle drawn with the foot of the perpendicular as center. **XXI**

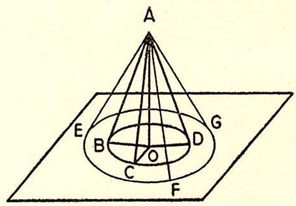

Fig. 26

There may be any number of such circles (their radii will be different), but this does not affect the property just stated (see Fig. 26). Incidentally, this property will be encountered when we study cones. (See page 344.)

TEST YOUR KNOWLEDGE OF LOCI IN SPACE WITH THESE EXERCISES

73 How would you locate a point equidistant from the front wall, a side wall, and the floor of a room? How many points are there for one corner?

74 Where are located in space all the points equidistant from two ends of a straight line-segment? (*Hint*: Refer to plane geometry.)

75 Where are all the points equidistant from the floor and the ceiling of a room located?

76 Locate a point in a room 3 feet from the floor and 5 feet from a wall.

77 A room is 9 feet high. A certain point is taken on the ceiling. What is the trace of a point on the floor such that its distance from the point on the ceiling is always 15 feet? What is the length of this path?

Polyhedral angles

When three planes, L, M, and N, intersect in a point, P (Fig. 27), they form several angles. The one which is included by the three planes is called a *trihedral angle*. The point where the three planes intersect is called the *vertex* of the trihedral angle. The three planes are called the *faces* of the trihedral angle. Note also that every two adjacent planes form a dihedral angle. A trihedral angle has: one vertex, three faces, and three edges (which are also the edges of the dihedral angles). Each face of a trihedral angle is included between two edges. These two edges form an angle, known as the *face angle* of the trihedral angle.

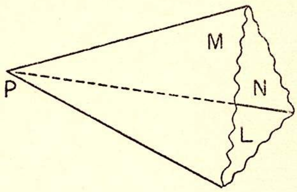

Fig. 27

More than three planes can intersect in one point. If four planes intersect in this manner, we obtain an angle with four faces, four edges, and one vertex. Such an angle is called a *tetrahedral angle*. Each face of a tetrahedral angle is of the form of an angle. This angle is formed by two adjacent edges (Fig. 28).

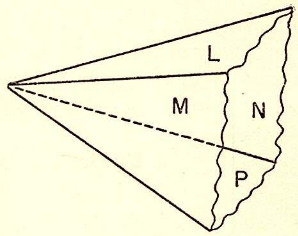

Fig. 28

The size of any polyhedral angle depends on the size of its face and on the order in which these angles are arranged.

TEST YOUR KNOWLEDGE OF POLYHEDRAL ANGLES WITH THESE EXERCISES

78 Draw angles of 60°, 80°, and 90°, as shown in Fig. 29. Cut the figure out along the edges, OA and OB, and fold along OC to form a trihedral angle. Now draw the same angles, but in the order, 80°, 60°, and 90°, and follow the instructions given above. The two trihedral angles so obtained will be different.

79 Draw angles of 45°, 60°, 90°, and 120° in the same manner as in Fig. 29. Then draw the same angles, but in some different order. Cut the figures out as explained above and fold them to form tetrahedral angles. If the orders selected are different, you will obtain two different tetrahedral angles.

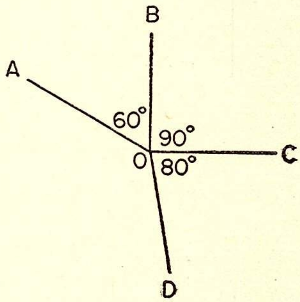

Fig. 29

80 Draw three angles (50°, 60°, and 90°) twice, and in the same order as shown in Fig. 29. Cut the two figures and fold them. Try to fit the two trihedral angles so obtained. What results can you observe?

81 State a generalized property based on the results obtained in Problems 79 and 80.

82 Try to obtain a tetrahedral angle by cutting four individual angles (100°, 90°, 120°, and 140°) and fitting them together. Can you explain the results obtained?

SOLID FIGURES In plane geometry, all the figures (excepting points, lines, and angles) studied were such that they enclosed some portion of a plane. A similar situation obtains in solid geometry. Excepting points, lines, planes, and all sorts of angles, the figures that are the subject of study in solid geometry are such that they enclose some portion of space separating that portion from the remaining space. The enclosed portion of space forms a solid figure. The enclosed portion is separated by planes or by some other kinds of surfaces, but, whatever the separating surfaces may be, they enclose the solid completely.

If the enclosing surfaces of a solid figure are planes and planes only, such solid figures are called *polyhedra*. There are many kinds of these polyhedra. They can be classified according to their properties and characteristics. The classification of polyhedra is usually made according to the number of faces there are in the solid figure, according to the relation between the faces (whether they are parallel or perpendicular to one another), according to the shapes of the faces,—that is, what kind of plane geometric figures they are. The most common polyhedra which we come to know are the cube, the pyramid, and a box-like solid which is called a parallelepiped.

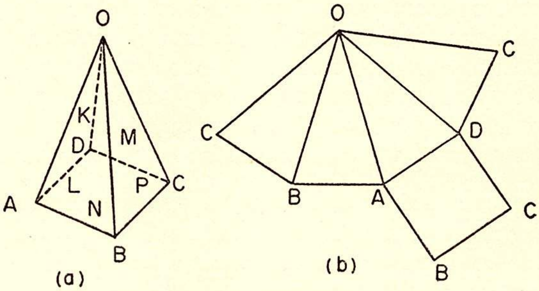

Fig. 30

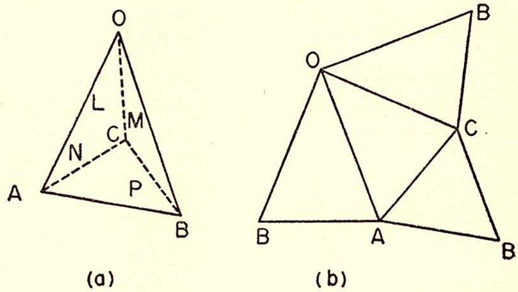

Fig. 31

Pyramids

If we take a polyhedral angle, $OABCD$, and pass a plane, P, that cuts all of its faces, the passed plane encloses the space within the polydedral angle (Figs. 30a and 31a). The figure so obtained is called a *pyramid*. There may be all sorts of pyramids. They are classified according to the number of faces they have. However, here a distinction is introduced. The face that is obtained by passing the plane as described above is not called a face; it is called the *base* of the pyramid. However, this is only a convenience which will become more obvious presently.

The smallest number of bounding planes that can enclose a portion of a space and form a solid figure is four.

If we take a trihedral angle, $OABC$, and pass a plane, P, as described, we obtain a pyramid with three faces and one base. All of them are triangles. Such a solid figure is called a *tetrahedron* (Fig. 31a), a figure with four faces (one of which is the base).

If a tetrahedron is cut along the three edges that come together (the mathematical term is *converge*), the faces and the base can be flattened out (Fig. 31b). This gives us a clue as to how a tetrahedron can be constructed.

If the base of a pyramid is a plane figure with more than three sides, the pyramid will have the same number of faces as there are sides in the base. Each face of a pyramid is a triangle.

If we split the pyramid open at all the edges at the base except one, and then split only one edge on the side, the pyramid's surface may be flattened out. This gives us a clue as to how a pyramid may be constructed generally (Fig. 31b).

There is one limitation in the construction of any pyramid: the bases of the face triangles must be equal to the corresponding sides of the polygon that forms the base of the pyramid.

TEST YOUR CONCEPT OF PYRAMIDS WITH THESE EXERCISES

83 Make drawings as shown in Figs. 30b and 31b. Cut them out and fold them to form pyramids.

84 Construct a pyramid with a five-sided base. First make a drawing.

85 Begin by making a drawing for the construction of a four-sided pyramid. Then draw, through the upper parts of all the triangles, straight lines so as to indicate that these upper parts are to be cut off. Continue by cutting out the drawing. Then cut off the upper parts of the triangles. Fold the figure into a solid and cover the portion that is left open. You will thus obtain a pyramid whose top has been cut off. This is known as a truncated pyramid. (Fig. 32).

THE LATERAL AREA OF A PYRAMID

Each face of a pyramid is a triangle. The formula for the computation of the area of a triangle, as obtained in plane geometry is

$$A=\frac{bh}{2}$$

where b is the length of the base, and h is the length of the altitude of the triangle. In order to obtain the total side area (called the *lateral area*) of a pyramid, we must obtain the area of each face, and add all these areas. The total sum is the lateral area of the pyramid.

Of all pyramids that are possible, those whose bases are regular polygons and whose edges (the sides of the triangular faces) are all equal are known as regular pyramids. Their regularity enables us to obtain simple formulas for the computation of their lateral areas (and volumes, as will be shown on page 348).

When we are computing the lateral area of a regular pyramid, we do not have to obtain the area of every triangular face. Since the sides

of a regular polygon are all equal and since the edges of a regular pyramid are all equal, the triangular faces of a regular pyramid are all congruent triangles, and their areas are all equal. Thus, in order to compute the lateral area of a regular pyramid, we need to compute the area of one face only. After this is accomplished, we multiply this area by the number of faces; and the computed product is the lateral area of the regular pyramid. Note that the formula for this lateral area becomes

$$S=n \cdot A \qquad \textbf{XXII}$$

where n is the number of faces, and A is the area of one triangular face.

Another important fact follows from our observation of regular pyramids. Since the faces of a regular pyramid are all congruent triangles, the various corresponding parts of these triangles are equal. **XXIII**

Thus, the altitudes of these triangles are equal. **XXIV**

It follows that the formula for the lateral area of a regular pyramid can be written as

$$S=\frac{nbh}{2} \qquad \textbf{XXV}$$

where n represents the number of faces, b the length of the side of the base, and h the lateral altitude (the altitude of a face triangle). It should be noted that, in a regular pyramid, the faces are all isosceles triangles since two of their sides are equal.

If a regular pyramid has its top cut off by a plane that is parallel to the base, we obtain a regular truncated pyramid (Fig. 32). The sides of this pyramid are trapezoids—particularly, isosceles trapezoids. In order to obtain the lateral area of a regular truncated pyramid, we obtain the area of one face and multiply this area by the number of faces. The formula for the area of a truncated pyramid is

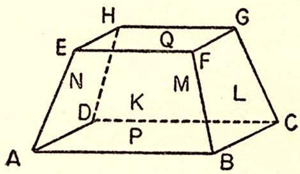

Fig. 32

$$A=\frac{h(a+b)}{2} \qquad \textbf{XXVI}$$

where h is the altitude of the pyramid, and a and b are the sides of the upper and lower bases respectively, of the truncated pyramid. The formula for the lateral area of a regular truncated pyramid becomes

$$S=\frac{n \cdot h(a+b)}{2}. \qquad \textbf{XXVII}$$

Note that this is a more general formula. From it, we can obtain the formula for the lateral area of a regular pyramid. All that we have to do is make $a=0$ because at the vertex of any pyramid we have a point.

TEST YOUR ABILITY TO MEASURE PYRAMIDS WITH THESE EXERCISES

86 Two pyramids have as their bases similar polygons. Their faces are all similar triangles (that is, the corresponding faces are similar). The ratios of the sides are all 1 : 3. In what ratios are the lateral areas of the two pyramids?

87 Derive a general statement for the problem stated in 86.

88 What is the lateral area of a regular pyramid whose base is a square with a side of 10, and whose edges are all 15?

89 What is the lateral area of a truncated regular pyramid obtained from a regular tetrahedron with all edges 10 after the upper part of the tetrahedron was cut off by a plane parallel to the base so that the three edges were cut in half.

The cube

A *cube* is a geometric solid with six faces, all of them squares. The opposite faces are parallel to one another. The adjacent faces are perpendicular to one another.

The *surface area* of a cube is obtained as the sum of the area of the six faces. Since the six faces of the cube are all squares, and the sides of these squares are all equal, the surface area of the cube is obtained as the product of the area of one face (square) by the number of these faces (six). We have, then,

$$S=6A \qquad \textbf{XXVIIIa}$$

where S is the surface area of the cube, and A is the area of one face (square).

The area of a square whose side is a is $A=a^2$. We have, therefore, for the surface area of the cube whose edge is a,

$$S=6a^2 \qquad \textbf{XXVIIIb}$$

For example, the surface area of a cube whose edge is 5 is $6{\cdot}5^2=6{\cdot}25=150$.

If we join the opposite vertices of a cube by a straight line, we obtain the diagonal of the cube (Fig. 33). The length of this diagonal can be computed if we recall the fact that a straight line perpendicular to a plane is perpendicular to all straight lines drawn in that plane through the foot of the perpendicular.

We have, then, a right triangle, ABC, in which AC, the diagonal of the cube is the hypotenuse. Then

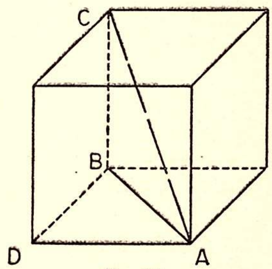

Fig. 33

$$\overline{AC}^2=\overline{AB}^2+\overline{BC}^2 \qquad \textbf{XXIX}$$

We know that BC is the edge of the cube. Let $BC=a$.

So far, AB is not known, but AB is the diagonal of the square that is the face of the cube. Then, from the right triangle, ABD, the value of the hypotenuse, AB, is obtainable from the relation,

$$\overline{AB}^2=\overline{AD}^2+\overline{BD}^2.$$

Since all the edges of a cube are equal, we have

$$\overline{AB}^2 = a^2 + a^2 = 2a^2.$$

Then
$$\overline{AC}^2 = 2a^2 + a^2 = 3a^2,$$

and
$$AC = \sqrt{3a^2} = a\sqrt{3} = 1.7321a.$$

Thus, the diagonal of a cube whose edge is 10 is 17.321.

TEST YOUR KNOWLEDGE OF CUBES WITH THESE EXERCISES

90 What is the surface area of a cube whose edge is 15?
91 What is the length of the diagonal of a cube whose diagonal of a face is 10?
92 What is the surface area of a cube whose diagonal is 3?
93 How many vertices are there in a cube?
94 How many edges are there in a cube?
95 How many diagonals are there in a cube?
96 What is the area of the diagonal plane of a cube whose edge is a? 5?
97 What is the surface area of a cube if the area of the diagonal plane is 10?

Prisms

A polyhedron whose upper and lower bases are congruent polygons (that is, equal in area and shape) lying in parallel planes is called a *prism*. All the lateral faces of a prism are parallelograms (Fig. 34). A prism is a general class of polyhedra to which the cube belongs. We shall examine the other polyhedra that belong to this class presently.

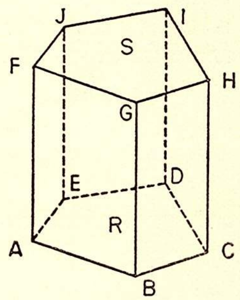

Fig. 34

It can be observed that, since all the lateral faces of a prism are parallelograms, the sides of these parallelograms which are also the lateral edges of a prism are all parallel and equal to one another. This is another characteristic of a prism.

If the lateral edges of a prism are perpendicular to the bases, the lateral faces (the planes) are also perpendicular to these bases. Such a prism is called a *right prism* (from the fact that the dihedral angles formed by the lateral edges and the bases are all right angles).

In a right prism, all the lateral faces are rectangles (Fig. 35). This follows from the facts stated above.

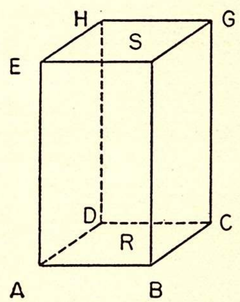

Fig. 35

If, in any prism (right or oblique), a plane is passed that is perpendicular to the lateral edges (and, therefore, to the lateral faces), we have a means for computing the lateral area of the prism (Fig. 36). Note that the intersections of this perpendicular plane are all lines that are perpendicular to the edges of the prism (and these are also the opposite sides of the parallelograms, the lateral faces of the prism). From plane geometry, we know that the area of a parallelogram is equal to the product of two numbers representing the length of a side and the altitude of the

parallelogram drawn to that side. The intersections of the perpendicular plane with the lateral faces are all altitudes of the parallelograms. Thus, we arrive at a rule for the computation of the lateral area of a prism: we add the lengths of all these perpendiculars (the sum is also known as the perimeter of the right section) and multiply this sum by the length of a lateral edge (remembering that all lateral edges are equal). This can be represented by the formula,

$$S=p \cdot e \qquad \text{XXX}$$

where S is the lateral area, p is the perimeter of the right section (the sum of all the perpendiculars), and e is the length of a lateral edge.

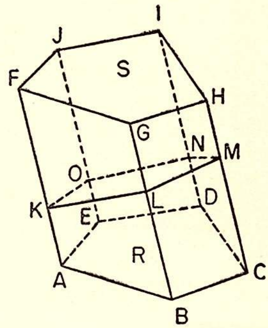

Fig. 36

In a right prism, it is not necessary to draw a perpendicular plane (the right section) because the bases of a right prism are already perpendicular to the lateral edges and lateral faces. Thus, the formula for the lateral area of a right prism remains the same, but p in this case is the perimeter of the base.

Prisms are generally classified according to the shapes of their bases. Thus, if the base is a triangle, we have a triangular prism. If the base is a four-sided figure, we have a quadrangular prism. If the base is a five-sided figure, we have a pentagonal prism.

TEST YOUR KNOWLEDGE OF PRISMS WITH THESE EXERCISES

98 In a right triangular prism, all the edges and sides of the bases are equal. The lateral area of this prism is twelve square inches. What is the length of an edge?

99 A right rectangular prism has a square base. The diagonal of the prism is 9 in. and the lateral area of the prism is 108 sq. in. Find the lengths of the side of the base and of the edge.

100 Find the total area of the surface of a triangular right prism whose edge is 50, the lengths of the sides of its base being 40, 13, and 37.

101 In a right triangular prism, the lengths of the sides of the base are in the ratios 17: 10: 9. The edge is 16 in. The total area of the surface of the prism is 1440 sq. in. Find the lengths of the sides of the base.

102 A prism (any, oblique or right) is cut by a plane parallel to an edge. What is the shape of the plane figure enclosing the surface in the intersecting plane?

PARALLELEPIPEDS

A quadrangular prism whose bases are parallelograms is called a *parallelepiped*. A cube is a special type of parallelepiped. There are other kinds of parallelepipeds, but all of them share certain properties which we shall examine. A parallelepiped (Fig. 37) has six faces (two

of which are the bases). If the lateral edges of a parallelepiped are perpendicular to the bases and the bases are rectangles, it is called a *rectangular parallelepiped* (a box-like solid figure), as shown in Fig. 38.

A cube is a parallelepiped all of whose faces are squares. Its lateral edges are perpendicular to the bases. **XXXI**

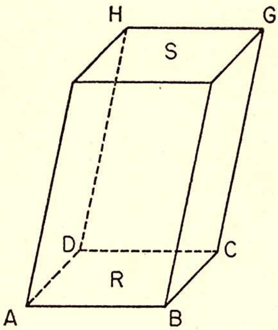

Fig. 37

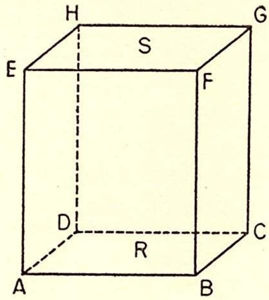

Fig. 38

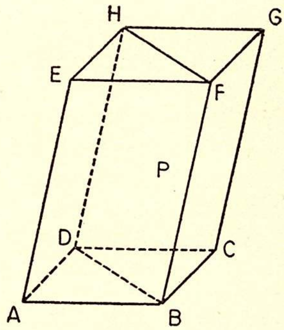

Fig. 39

In a parallelepiped, the opposite faces are parallelograms equal in area and of the same shape. **XXXII**

This property follows from a fact that can be observed by reverting to plane geometry. (This is left for the reader as an exercise.)

All other properties of prisms apply equally to parallelepipeds; parallelepipeds are prisms.

A diagonal plane, P, drawn through a parallelepiped, *ABCD-EFGH* (Fig. 39) cuts the solid figure into two equal triangular prisms, *ABD-EFH* and *BDC-FHG*.

TEST YOUR KNOWLEDGE OF PARALLELEPIPEDS WITH THESE EXERCISES

103 A lateral edge of an oblique prism is 8 in. long. The distances between the consecutive lateral edges are 3 in., 6 in., 2 in., and 7 in. (a) What is the lateral area of the prism? (b) Is the base of the prism a parallelogram?

104 (a) Can there be a parallelepiped whose bases are squares and only two of whose lateral faces are squares? (b) If so, will such a solid be a rectangular parallelepiped?

105 The base of a parallelepiped is a square. One of the upper vertices of the parallelepiped is equally distant from all the vertices of the lower base. The side of the base is b and the lateral edge is a. What is the area of the total surface of the parallelepiped?

106 What are the lengths of the diagonals and the areas of the diagonal sections of the parallelepiped in Problem 105?

107 What is the shape of the faces of a rectangular parallelepiped?

108 The lateral edge of a rectangular parallelepiped is 10 in. The lengths of the sides of the bases are 5 in. and 8 in. What are the lengths of the diagonals of the parallelepiped and what are the areas of the diagonal sections?

Cylinders

Solid figures may be bounded entirely by surfaces other than planes, or they may be enclosed partly by planes and partly by a surface or surfaces other than planes. A certain group of such solid figures, which resemble prisms, may have as their bases closed curved figures (whose planes are parallel and whose shapes and areas are equal). The lateral

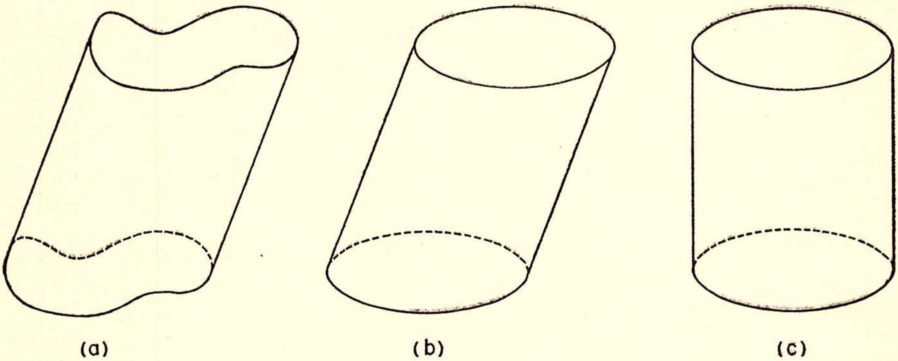

Fig. 40

surface of such a solid is not a combination of planes; it is curved. Such a solid, cylindrical in shape, is called a *cylinder* (Fig. 40a).

Of special interest is a cylinder whose two bases are circles. Such cylinders may be oblique (Fig. 40b) or right (Fig. 40c). In a right cylinder, the lateral surface is perpendicular to the bases.

A cylindrical surface may be thought of as formed (or, as mathematicians say, generated) by moving a straight line always parallel to itself along the curved outline of the base (Fig. 41). Thus, on a cylindrical surface, a straight line, AB, will lie with all of its points in one direction. This distinguishes a cylindrical surface from a plane surface. The straight line generating the cylindrical surface is called its *generatrix*.

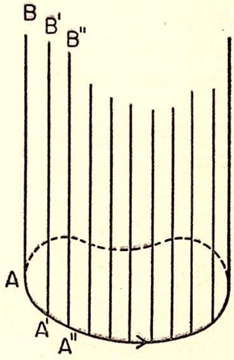

Fig. 41

If the cylinder is cut by a plane, and this plane contains within itself a line that is parallel to the generatrix, then the section of the cylinder is a parallelogram, as shown in Fig. 42. If the cylinder is a right one, this section is a rectangle.

The rule concerning the lateral area of a cylinder follows the lines of the rule for the lateral area of a prism. If the cylinder is an oblique one, we must obtain a section that

is perpendicular to a line parallel to the generatrix (known as the *element* of the cylinder), and the lateral area is obtained as the product of the length of this section, as outlined on the surface of the cylinder (Fig. 43) and of the length of the element. If the cylinder is a right one, the lateral area is obtained as the product of the element by the circumference of the base.

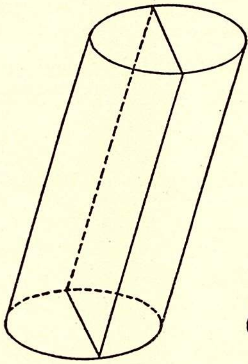

Fig. 42

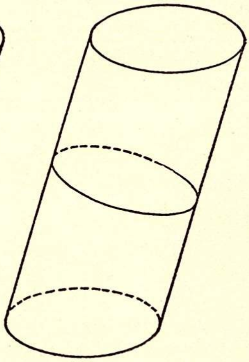

Fig. 43

Illustrative Example

The lateral area of a right cylinder whose base is a circle is computed as follows:

Let the radius of the base be $r=5$ in., and the length of an element be $h=10$ in.

The lateral area of the cylinder is then

$$S=h\cdot p \qquad \text{XXXIII}$$

where p is the length of the base.

The base is a circle whose radius is r. Then

$$p=2\pi r=2\cdot 3.14r=6.28r.$$

Then $$S=2\pi r\cdot h, \qquad \text{XXXIV}$$

which is the general formula for the lateral area of a cylinder. If we substitute the values of r and h, we obtain

$S=6.28\cdot 5\cdot 10=314$ sq. in.

A right circular cylinder (whose bases are circles) may be thought of as having been generated by a rectangle that was revolved around one of its sides as an axis, as shown in Fig. 44. Thus, such a right circular cylinder may be called a *cylinder of revolution*.

If two similar rectangles are revolved in this fashion around corresponding sides, we obtain two similar cylinders of revolution.

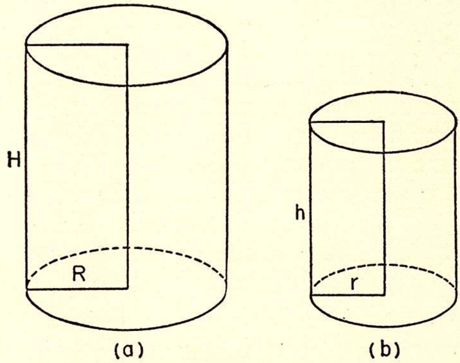

Fig. 44

In two similar plane figures (Fig. 44a and 44b), the corresponding sides are proportional. XXXV

From this fact we obtain the following important property of the lateral areas of two similar circular right cylinders:

$$\frac{r}{R}=\frac{h}{H}=k \text{ (some number)},\ S_1=2\pi rh,\ S_2=2\pi RH$$

$$\frac{S_1}{S_2} = \frac{rh}{RH} = k^2$$

But
$$k^2 = \left(\frac{r}{R}\right)^2 = \left(\frac{h}{H}\right)^2.$$

In other words, the lateral areas of similar right circular cylinders are proportional to the squares of their corresponding parts (radii of bases, or elements). **XXXVI**

TEST YOUR KNOWLEDGE OF CYLINDERS WITH THESE EXERCISES

109 What is the total area of a right circular cylinder whose element (also known as the altitude) is 15 ft. and the radius of whose base is 4 ft.?

110 If the altitude of a right cylinder is kept unchanged, how should the radius of the base be changed so that the lateral area of the cylinder be (a) doubled, (b) tripled?

111 What is the shape of a plane section of a right cylinder if the section is taken perpendicular to the element?

112 If the radius of the base of a right cylinder is kept unchanged, how should the altitude of the cylinder be changed so that the lateral area be (a) doubled, (b) quadrupled?

113 A square whose side is 5 is revolved around one of the sides. What is the total area of the solid of revolution?

Cones

If we take a point, *A*, outside a plane, *P*, and draw through this point an oblique line, *AB*, to the plane, fixing the length of the line and revolving it so that it traces out a curve on the plane (this curve is a circle), we obtain at the same time a surface that is traced by the straight line while it revolves (Fig. 45). This surface is known as a *conical surface.* This cone has two parts or *nappes*, converging at the point, *O*, which is called the *vertex*.

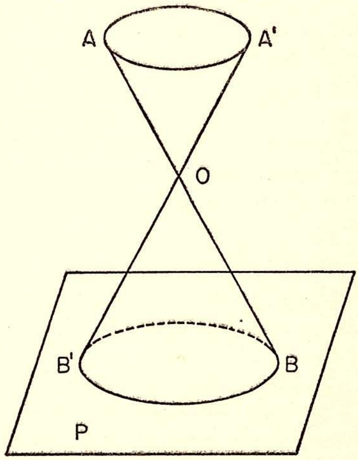

Fig. 45

A conical surface need not have a circular base, as is shown in Fig. 45. The base may be of any shape, but it must be a closed curved line (such as an ellipse, oval, or any other shape). The surface is conical as long as the surface converges into a point (in a manner similar to the case of a pyramid).

The revolving straight line is known as the *generatrix* of the cone.

If the cone has a circular base and the point on the top of the cone is exactly above the center of the circle of the base (that is, the line joining the vertex and the center is perpendicular to the plane of the base), the

cone is known as a *right circular cone.* In a right circular cone, the generatrix is of the same length for the entire cone. The generatrix also acts in the same manner as the altitude of the face triangle of a pyramid, as shown in the next paragraph.

MEASUREMENT OF CONES

If either nappe of a right circular cone is slit along some position of the generatrix, the lateral surface of the cone can be flattened out as shown in Fig. 46. This enables us to obtain the rule for the computation of the lateral area of a right circular cone. We may think of the flattened out lateral surface as of a triangle. The generatrix of the cone is then the altitude of the triangle. The base of the triangle is the lower edge of the cone, and its length is equal to the length of the circumference of the circular base. We then have the formula,

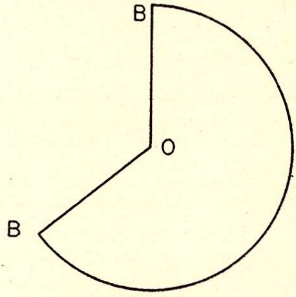

Fig. 46

$$S=\frac{1}{2}\cdot 2\pi r\cdot l,$$

or

$$S=\pi r\cdot l. \qquad \textbf{XXXVII}$$

Some call the generatrix of the right circular cone its *slant height.*

For example, the lateral area of a right circular cone with a base whose radius is 10 in. and whose slant height is 15 in. is

$$3.14\cdot 10\cdot 15=471 \text{ sq. in.}$$

If the upper part of a right circular cone is cut off by a plane parallel to the base, the section is circular in shape, and the new solid figure is called a *frustum* of a right circular cone (Fig. 47). The frustum of a cone is somewhat akin to the frustum of a pyramid. The formula for the lateral area of a frustum of a right circular cone is

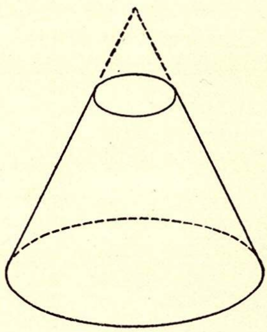
Fig. 47

$$S=\frac{1}{2}\cdot 2\pi(r+R)\cdot l=\pi(r+R)\cdot l. \qquad \textbf{XXXVIII}$$

For example, the radii of the upper and lower bases of a frustum of a right circular cone are 5 and 15 inches respectively, and the slant height of the cone is 6 inches. The area of the lateral surface of the cone is $3.14(5+15)6=(3.14)\cdot 20\cdot 6$
$=376.8$ sq. in.

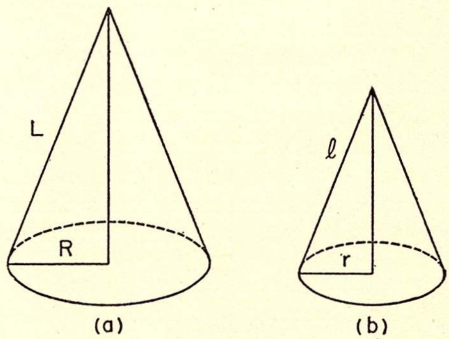

Fig. 48

A right circular cone may be thought of as generated by a right triangle that was revolved about one of its sides (not the hypotenuse) as an axis. If two similar right triangles are so rotated about their

corresponding sides, we obtain two similar right circular cones. From the fact that the corresponding sides of similar plane figures are proportional, we may obtain a relationship between the lateral areas of two similar right circular cones. Fig. 48 shows that

$$\frac{r}{R}=\frac{l}{L}=k. \qquad \text{XXXIX}$$

We already know that the respective lateral areas of the two right circular cones are

$$S_1=\pi r \cdot l \text{ and } S_2=\pi R \cdot L$$

and

$$\frac{S_1}{S_2}=\frac{r \cdot l}{R \cdot L}=k^2$$

Now

$$k^2=\left(\frac{r}{R}\right)^2=\left(\frac{h}{H}\right)^2.$$

$$\therefore \quad \frac{S_1}{S_2}=\left(\frac{r}{R}\right)^2=\left(\frac{h}{H}\right)^2,$$

that is, in two similar right circular cones the lateral areas are in the ratio of the squares of the corresponding elements (radii of the bases or the slant heights). XL

TEST YOUR KNOWLEDGE OF CONES WITH THESE EXERCISES

114 What is the lateral area of a right circular cone whose slant height is twice as long as the radius of the base? The circumference of the base is 125.6 in.

115 What is the shape of the section of a right circular cone that is perpendicular to the base and passes through the center of the base?

116 How does the lateral area of a right circular cone change if the slant height is kept unchanged while the radius of the base is doubled?

117 How does the lateral area of a right circular cone change if the radius of the base is kept unchanged while the slant height is tripled?

118 Obtain the formula for the total area of the surface of a right circular cone.

119 How does the total area of the surface of a right circular cone change if the slant height is doubled and the radius of the base is also doubled?

A FINAL CHECK ON AREAS OF SURFACES

120 The area of the largest diagonal section of a regular right prism with a hexagonal (six-sided base) is 1 sq. ft. What is the lateral area of the prism?

121 The distances between the lateral edges of an oblique triagonal prism are 2 in., 3 in., and 4 in. The lateral area is 45 sq. in. What is the length of the lateral edge?

122 In an oblique triagonal prism, the length of an edge is 8 in. The sides of a perpendicular section are in the ratios 9:10:17. The area of the perpendicular section is 144 sq. in. What is the lateral area of the prism?

123 In a regular quadrangular pyramid, the side of the base is a. The base is a square. The area of the diagonal section is equal to the area of the base. What is the lateral area of the pyramid?

124 What is the lateral area of a regular hexagonal pyramid if the side of the base is a and the area of one face is equal to the area of the largest diagonal section?

125 In a frustum of a regular triangular pyramid, the dihedral angle at the base is 60°. The side of the base is a, and the total area is S. What is the length of the side of the other base?

126 In a frustum of a regular quadrangular pyramid, the area of the bases are A and B, and the lateral area is C. What is the area of the diagonal section?

127 A cylindrical boiler is 0.7 yards in diameter. Its length is 3.8 yards. What is the total pressure on the entire surface of the boiler if the pressure on every square inch is 0.25 lbs.?

128 In steam heating, the standard is that every square yard gives out about 550 heat units every hour. What should be the length of the pipe 1 inch in diameter so that the heat given out will be 4500 units every hour?

129 R is the radius of the base of a right circular cylinder. The lateral area of the cylinder is equal to the sum of the areas of its bases. What is the length of its generatrix?

130 A tent of the shape of a right circular cone is 3.5 yards high. The radius of the base is 4 yards. How much canvas 1 yard wide is needed?

131 A right triangle whose sides are 5 in. and 12 in. long is revolved about its hypotenuse. What is the total area of the solid of revolution?

132 A semicircle is folded into a right circular cone. What is the radius of the base of the cone? The radius of the semicircle is r.

133 A draft flue is of the form of a frustum of a cone. The diameter of the top is 0.2 yards. The diameter of the lower part is 1 yard. The slant height is 0.7 yards. How much sheet iron is needed for this flue?

134 The areas of the bases of a frustum of a circular cone are A and B. What is the area of a section parallel to the bases and equidistant from them?

Volumes

A general formula that can be conveniently employed for the computation of the volumes of the common solid figures is

$$\frac{h(B_1+B_2+4M)}{6} \qquad \text{XLI}$$

(Fig. 49.) This formula is so general that it can be applied to the computation of areas of plane figures as well. The reader may check this statement by obtaining the formulas for the areas of various plane figures as for the trapezoid, for instance, in which we multiply the sum of the bases by the altitude and divide by 2:

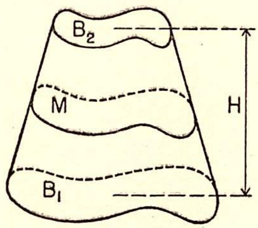

Fig. 49

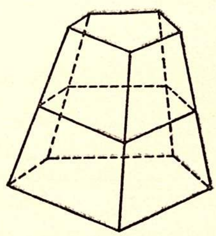
Fig. 50

$$\frac{h(B_1+B_2)}{2}$$

In the process of the derivation of the formulas for volumes of various solids, we shall make use of the following properties of sections drawn parallel to the base of any solid figure: if a plane is passed so that it is parallel to the base of a solid figure and it cuts the edges of the solid figure, then the section so obtained is a plane figure similar to the base (Fig. 50). **XLII**

The property stated in **XLII** can be applied in the process of using formula **XLI**. Here we must also recall the fact that areas of similar plane figures are proportional to the squares of their corresponding sides or elements (such as altitudes). The reader should also refer to the treatment of similar figures in plane geometry.

Prisms

The volume of an oblique prism is equal to the volume of a prism whose base is a section perpendicular to the lateral edge of the prism and whose altitude is the lateral edge of the first (oblique) prism.

The new prism is a right prism, as can be observed in Fig. 51a.

Note that, in any prism, the bases are parallel to one another.

Furthermore, these bases are equal (and congruent). When a perpendicular section is passed in a prism, the prism may be thought of as having been cut into two parts.

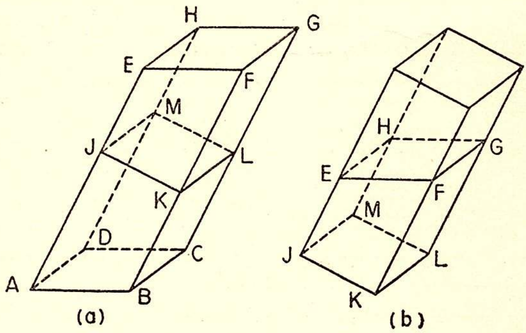

Fig. 51

Now, if we lift one part of the prism and fit the upper and lower bases of the oblique prism onto one another, we obtain a right prism (Fig. 51b). This establishes the fact stated above.

CUBES

The volume of a cube is equal to the product of the area of the base by the side of the cube. This follows directly from the property stated in **XLII** above. The base of the cube is perpendicular to the lateral edges. Thus, we have

$$V_{cube} = h \cdot A. \qquad \textbf{XLIII}$$

Since h is equal to the side edge of a cube (which is, say a) and A is the area of a square (whose side is also a),

$$A = a^2.$$

Finally,

$$V_{cube} = a \cdot a^2 = a^3. \qquad \textbf{XLIV}$$

This is the formula for the volume of a cube.

PARALLELEPIPEDS

The volume of a rectangular parallelepiped is equal to the product of its three dimensions—that is, width, length, and height. This follows from the fact stated in **XLII**. The rectangular parallelepiped is also a rectangular prism whose base is a rectangle (Fig. 51). The volume of this solid figure is

$$V_{rect.\ parallelepiped} = h \cdot A. \qquad \textbf{XLV}$$

The area, A, of the rectangular base is obtained as the product of the two dimensions of the base (width $= w$, and length $= l$), that is

$$A = w \cdot l.$$

Then

$$V_{rect.\ parallelepiped} = h \cdot w \cdot l. \qquad \textbf{XLVI}$$

PYRAMIDS

In order to obtain the formula for the volume of a pyramid, we must use the formula stated in **XLI**. Furthermore, we shall recall the fact that a line drawn from the mid-point of the side of any plane triangle parallel to a second side is equal to one-half of the length of the second side.

If a plane is passed midway between the base and the vertex of a pyramid, it divides all the lateral edges in two.

Each lateral face of a pyramid is a triangle.

Thus, the intersection of the passed plane and a face is a straight line, and the length of that straight line is equal to one-half of the base of the pyramid that belongs to that lateral face. Furthermore, considering the property stated in **XLII**, the plane section is a plane figure similar to the base of the pyramid.

Since the sides of the plane section and of the base are in the ratio of 1 : 2, their areas are in the ratio of 1 : 4 (since $2^2 = 4$). Then, in the formula stated in **XLI**,

$$4M = B.$$

The area at the vertex of a pyramid is zero, that is $A = 0$. We have then, after substituting these values in the formula given in **XLI**,

$$V_{pyramid} = \frac{h(0+B+B)}{6} = \frac{2hB}{6} = \frac{hB}{3}, \qquad \textbf{XLVII}$$

which is the formula for the volume of a pyramid.

CONES

The formula for the volume of a right circular cone, or any other cone with a circular base, or a cone with any other curved base is obtained in the same manner as the volume of a pyramid. The area of a midsection of such a cone is similar to the area of the base. The

relation of the area of the midsection to the area of the base is again

$$4M=B$$

and the area at the vertex of the cone is zero. We then obtain, by the same process as in **XLVII**, that the formula for the volume of a cone is

$$V_{cone}=\frac{Bh}{3}. \qquad \text{XLVIII}$$

If the base of the cone is a circle whose radius is r, the volume of the cone is

$$V_{cone}=\frac{\pi r^2 h}{3}. \qquad \text{XLIX}$$

If the base of a cone is an ellipse with semi-axes a and b (the area of the ellipse is πab), the volume of the cone is

$$V_{cone}=\frac{\pi abh}{3}. \qquad \text{L}$$

The volume of a cylinder is obtained in a manner similar to the process of obtaining the formula for the volume of a prism. We can adjust our procedure, however, and dispense now with the necessity of obtaining the right section. The distance between the bases of the cylinder (and it may just as well be applied to the prism) is measured along a perpendicular, which is called the altitude of the solid figure (h). If we apply the formula given in **XLI**, we note that the area of the midsection as well as the areas of the two bases are all equal. That is,

$$A=M=B.$$

Substituting in the formula given in **XLI**, we have

$$V_{cylinder}=\frac{h(B+4B+B)}{6}=hB. \qquad \text{LI}$$

If the base of the cylinder is a circle whose radius is r, the area of $B=\pi r^2$, and

$$V_{cylinder}=\pi r^2 h. \qquad \text{LII}$$

If the base of the cylinder is an ellipse whose semi-axes are a and b, the area, $B=\pi ab$, and

$$V_{cylinder}=\pi abh. \qquad \text{LIII}$$

TEST YOUR KNOWLEDGE OF VOLUMES WITH THESE EXERCISES

135 The volume of a cube is 8 cu. ft. What is the area of its total surface?

136 Three metal cubes whose edges are 3 in., 4 in., and 5 in., are melted into one cube. What is the edge of the new cube?

137 If the edge of a cube is lengthened by 2 in., its volume is increased by 98 cu. in. What is the length of the edge of the original cube?

138 The surface of a cube (in square units) and the volume of the cube (in cubic units) are both expressed by the same number What is the length of the edge of the cube?

139 The diagonal of a rectangular parallelepiped is 35 in. The edges are in the ratios 2:3:6. What is the volume of the parallelepiped?

140 The cross-section of a railroad bed is of the form of an isosceles trapezoid whose bases are 42 ft. and 24 ft., and the side is 10 ft. How many cubic feet of earth are there in 1 mile of railroad bed?

141 What is the volume obtained by revolving a square of side a about one of its edges?

142 In what ratio are the volumes of two cylinders whose altitudes are the same but the areas of whose bases are not equal?

143 In what ratio are the volumes of two circular cylinders whose bases are equal but whose altitudes are not equal?

144 How should the radius of the base of a cylinder be increased so that the volume of the cylinder is doubled?

145 The lateral surface of a cylinder can be flattened out into a square whose side is a. What is the volume of the cylinder?

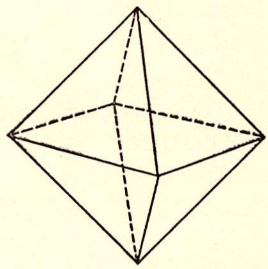

Fig. 52

146 The base of a prism is a triangle one of whose sides is 2 in., the other two sides being 3 in. each. The lateral edge of the prism is 4 in., and it makes a 45° angle with the base. What is the length of the edge of a cube whose volume is equal to the volume of the prism?

147 The altitude of a regular rectangular pyramid (the base is a square) is 3 in. The lateral edge is 5 in. What is the volume of the pyramid?

148 The edge of a regular octahedron (Fig. 52) is a. What is (a) the total surface and (b) the volume of the octahedron?

149 What portion of the volume of a pyramid will be cut off by a plane passed parallel to the base so that it is equidistant from the vertex and the base?

150 The altitude of a cone is 6 in., and the lateral surface of this cone is 24π sq. in. What is the volume of the cone?

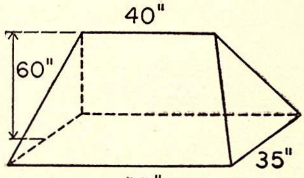

Fig. 53

151 The generatrix of a cone is a. The circumference of the circular base is C. What is the volume of the cone?

152 The generatrix, a, of a right circular cone makes and angle of 30° with the plane of the base. What is the volume of the cone?

153 The dimensions of a wedge are given in Fig. 53. The base of the wedge is a rectangle. Apply formula *XLI*, given in the text, and obtain the volume of the wedge.

154 Show by means of a cone or a pyramid that the intensity of illumination varies inversely with (is inversely proportional to) the square of the distance of the illuminated surface from the source of light. (*Hint*: Draw a cone or a pyramid and pass a plane parallel to the base so that the plane is equidistant from the base and the vertex. Then pass a plane parallel to the base at any distance from the vertex. Recall a property of the plane sections so obtained in relation to the base.)

SPHERICAL SURFACES

A *sphere* (or spherical surface) is a curved closed surface such that all of its points are at the same distance (otherwise stated, equally distant) from one point within the space enclosed by this surface. This point is called the *center* of the sphere. Any straight line-segment that joins the center of the sphere with a point on the surface is called the *radius* of the sphere. A straight line-segment drawn within the sphere and passing through the center is called the *diameter* of the sphere. A plane passing through the sphere intersects the sphere in a circle. If the plane passes through the center of the sphere, this circle has the same radius as the sphere and is known as the *great circle.*

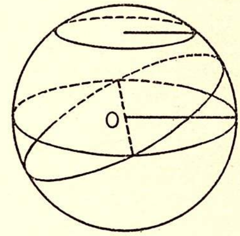

Fig. 54

Properties of spheres

It should be noticed that the plane of a great circle bisects (divides equally) the sphere (its surface as well as volume). Furthermore, two great circles on a sphere bisect one another. Fig. 54 illustrates all the properties just stated; the reader should refer to this figure when solving the exercises below.

The diameter of a sphere which is perpendicular to the plane of a circle of the sphere (passing through the center of that circle) is known as the *axis* of that circle. The points where this diameter intersects the surface of the sphere are known as the *poles* of that circle.

A spherical distance between two given points on a sphere is the length of the smaller portion of the great circle that passes through two points whose distance is considered (Fig. 55). The notion of the spherical distance on a sphere is a very important one, and the reader should take note of it.

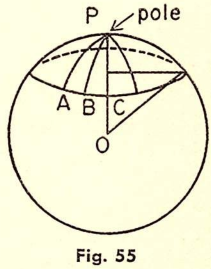

Fig. 55

It should be noted that the spherical distances, PA, PB, and PC, from points on a circle to the pole of this circle are all equal. (Fig. 55).

The reader can convince himself of this by considering the great circles in the figure and reverting to plane geometry, recalling that: (*a*) All great circles in the same sphere are equal; (*b*) the distance OP from the plane of the circle to its pole is the same for a given circle.

Here the reader should note that the distance along a sphere from a point on a circle to the pole of that circle is known as the *polar distance* of that point.

The properties of lines and circles learned in plane geometry are easily generalized for solid geometry. Usually the word, *circle*, in such property is replaced by *sphere*, and the word, *line*, is replaced by *plane*. Many of these properties are stated in the exercises below.

TEST YOUR KNOWLEDGE OF SPHERES WITH THESE EXERCISES

155 In how many points does a straight line cut a circle?

156 If a plane is tangent to a circle, in how many points does it touch it?

157 Is the radius of a sphere drawn to the point where a tangent plane touches the sphere perpendicular to the tangent plane?

158 How many straight lines can be drawn at a point on a sphere tangent to the sphere?

159 A point is taken outside of a sphere. How many straight lines can be drawn from that point tangent to the sphere? What kind of surface will all these straight lines form?

160 How many great circles can be drawn through one point of a sphere?

161 What is the polar distance of a point on a great circle?

162 The radius of a sphere is r. A plane section is made at a distance a from the center of the sphere. What is the area of this section? (*Hint*: Use the Pythagorean theorem.)

163 What is the intersection of two spheres? (*Hint*: Revert to plane geometry and draw two intersecting circles. Then consider the fact that a sphere may be obtained by revolving a semicircle about its diameter.)

164 What kind of circles are the meridians on the globe illustrating the earth's surface?

Measurement of spheres

In order to obtain the formula for the computation of the area of the surface of a sphere, we shall first compute the volume of a sphere whose radius is r. For this, we shall use the formula for volumes, **XLI**.

VOLUME

In this formula, h becomes the diameter of the sphere. Thus,

$$h=2r.$$

B_1 and B_2 are the upper and lower bases of the sphere. These are points. Thus,

$$B_1=B_2=0.$$

Finally, M is the area of the midsection of the sphere. This is the area of the plane of a great circle. Thus,

$$M=\pi r^2.$$

Substituting these values in the formula for the volume, we have

$$V_{sphere}=\frac{2r(0+4\pi r^2+0)}{6}=\frac{4\pi r^3}{3}. \qquad \textbf{LV}$$

For example, the volume of a sphere whose radius is 2 in. is

$$\frac{4(3.14)8}{3}=33.49 \text{ cu. in.}$$

AREA OF SURFACES

In order to obtain the formula for the area of the surface of a sphere, let us divide the volume of the sphere whose radius is r into a great number of pyramids of equal volume (Fig. 56).

Let the number of these pyramids be n. If this number is indefinitely great, we may (without introducing much error) consider the bases of these pyramids to be plane.

Let the area of each base be B.

The altitude of each pyramid is r.

Then the volume of each pyramid is

$$V=\frac{Br}{3}.$$

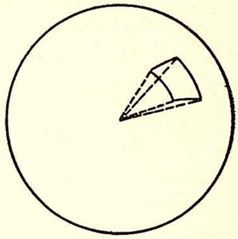

Fig. 56

Since there are n such pyramids in the sphere, the volume of the sphere is obtained as the product of the volume of one pyramid multiplied by n. Then

$$\frac{nBr}{3}=\frac{4\cdot\pi r^3}{3}.$$

From this, we have

$$nB=4\pi r^2,$$

and this, since nB is the product of the area of the base of one pyramid by the number of pyramids, is the area of the surface of the sphere. Thus, the formula for the area of the surface of a sphere whose radius is r is

$$A=4\pi r^2. \qquad \text{LVI}$$

TEST YOUR ABILITY TO MEASURE SPHERES WITH THESE EXERCISES

165 The diameter of a sphere is 25 in. On the surface of the sphere is given a point, A, and all the points whose distance (as measured by a straight line) from this point is 15 in. What is the radius of the circle to which these points belong?

166 The radius of a sphere is 15 in. Outside a sphere is a point, A, whose distance from the surface of the sphere is 10 in. What is the circumference of the circle whose points are 20 in. distant from A?

167 The radii of two spheres are 25 in. and 29 in., and the distance between their centers is 36 in. What is the length of their intersection?

168 The area of the surface of a sphere is 225π sq. in. What is the volume of the sphere?

169 The areas of the surfaces of two spheres are in the ratio $m : n$. In what ratio are their volumes?

170 The volume and the area of the surface of a sphere are expressed by the same number. What is the radius of the sphere?

171 A piece of metal in the form of a right cylinder whose altitude was equal to the diameter of the circular base was melted into the shape of a sphere. What was the change in the area of the surface of the solid object?

172 The volumes of two spheres are in the ratio of $m : n$. In what ratio are the areas of their surfaces?

173 The perimeter of the great circle of a sphere is 1 ft. What is the area of the surface of the sphere?

THE EQUATION OF A SPHERE

From the definition of a sphere (that its points are all equidistant from a point within the sphere, known as its center) we can obtain the expression for the surface of the sphere in terms of three coördinates and the radius of the sphere. No drawing is necessary.

Suppose that we have a point on the surface of the sphere with the coördinates (x, y, z).

If the radius of the sphere is r, then, from the property of the sphere, we have

$$x^2+y^2+z^2=r^2. \qquad \text{LVII}$$

This is the *equation of a sphere* when the center of the sphere is at the origin.

If the coördinates of the center of the sphere are (a, b, c), then the equation becomes

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2. \qquad \text{LVIII}$$

This is the *general equation* for the surface of the sphere.

Illustrative Example

If the coördinates of the center are (1, 2, 3), and the radius of the sphere is 5, then the equation of the sphere is

$$(x-1)^2+(y-2)^2+(z-3)^2=25.$$

The reader will notice that this is simply a generalization of the equation of the circle as obtained in plane coördinate geometry (Issue Number Five, page 296). The equation of the circle was

$$(x-a)^2+(y-b)^2=r^2.$$

Thus, the transition from plane to solid geometry involves only the tacking on of a third term containing the third coördinate, z.

THE EQUATION OF A CYLINDER

The expression, or equation,

$$(x-a)^2+(y-b)^2=r^2, \qquad \text{LIX}$$

has also a meaning in solid coördinate geometry. Note that the third coördinate, z, is missing—that is, $z=0$. In other words, the trace of the surface on the XOY plane is a circle. Furthermore, regardless of what value we might give to z, this section could be lifted and moved perpendicularly to the XOY plane while the plane of the section remains parallel to it. Thus, this equation is an equation of a cylinder, a right circular cylinder.

TEST YOUR KNOWLEDGE OF EQUATIONS WITH THESE EXERCISES

174 Write the equation of a sphere whose center's coördinates are (1, 0, 1) and whose radius is 10.

175 Write the equation of a right circular cylinder whose radius is 5, and the coördinates of the base's center are (0, 5, 3).

The ellipsoid

Let the equation of a sphere whose center is at the origin and whose radius is r be

$$x^2+y^2+z^2=r^2.$$

From this, we have

$$z=\pm\sqrt{r^2-x^2-y^2}.$$

Let us suppose that the value of z is uniformly contracted, that is, the vertical (z) coördinates are regularly made smaller, say, in the ratio, $\frac{c}{r}$. Then

$$z=\pm\frac{c\sqrt{r^2-x^2-y^2}}{r}.$$

Squaring and simplifying, we obtain

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{c^2}=1.$$

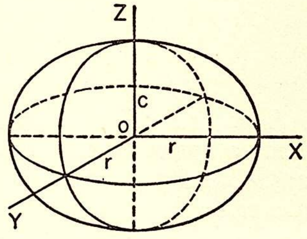

Fig. 57

This is an equation of a solid figure whose shape is no longer spherical; it is a *spheroid* (Fig. 57). The earth is of this shape. The horizontal section of this figure is a circle (along the plane, XOY). The sections along the other two planes are ellipses. This can be seen by making (one at a time) x, y, and z equal zero.

If we perform the contraction once more, by contracting, say, y in the ratio, $\frac{b}{r}$ we proceed as follows:

$$y=\pm r\sqrt{1-\frac{x^2}{r^2}-\frac{z^2}{c^2}}.$$

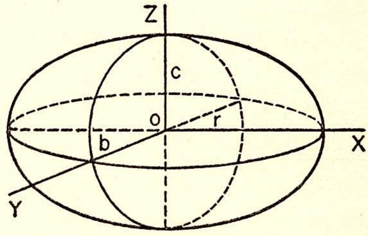

Fig. 58

Then

$$y=\pm\frac{b}{r}r\sqrt{1-\frac{x^2}{r^2}-\frac{z^2}{r^2}}.$$

From this, we obtain

$$\frac{x^2}{r^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \text{LXI}$$

This is the equation of a general *ellipsoid*. Its three sections are all ellipses (Fig. 58).

The values, r, b, and c, are the three semi-axes of the ellipsoid. Thus, if the three semi-axes are given, we can write the equation either of the ellipsoid or of the spheroid, depending on the values given.

Illustrative Example

If the three semi-axes are 3, 4, and 5, the solid is an ellipsoid, whose equation is

$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1.$$

TEST YOUR KNOWLEDGE OF SPHEROIDS WITH THESE EXERCISES

176 The earth is an oblate spheroid, the polar axis being about $\frac{1}{300}$ shorter than the equatorial axis (the diameter of the equator). Take the earth's center as the origin, and the polar axis as the OX-axis, and the two equatorial radii at right angles as the OY- and OZ-axes. Write the equation for the surface of the earth.

177 What are the equations of the sections on the coördinate planes of the solid,

$$4x^2+12y^2+5z^2=12?$$

What curves are they?

Angles on a sphere

Two arcs of great circles drawn from the same point, A, on a sphere form a *spherical angle*, DAE. This point is known as the *vertex* of the angle, and the arcs, AD and AE, of the great circle are called the *sides* of the spherical angle. This is shown in Fig. 59.

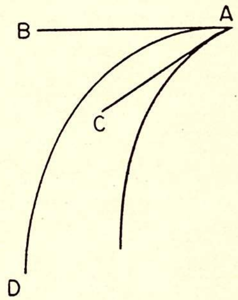

Fig. 59

The measure of a spherical angle is obtained by means of tangents, AB and AC, drawn to the sides at the vertex. Thus, the measure of a spherical angle is a plane angle. **LXII**

An angle that is formed by two arcs of great circles has the same measure as the arc of the great circle described from the vertex as a pole. If necessary, the sides should be produced to meet this great circle, as is shown in Fig. 60. The reader will note that the plane angle, BAC, and the angle formed by the two radii of the sphere, DOE, are parallel. The vertex of the spherical angle is the pole of the great circle—that is, AO is a radius perpendicular to the plane of the angle, DOE, and the plane of the angle, BAC, is perpendicular to the radius, AO.

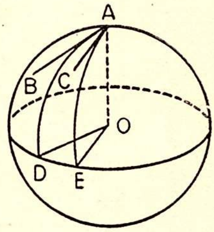

Fig. 60

Three arcs of great circles so drawn on a sphere that they form a closed figure form a *spherical triangle* (Fig. 61). Spherical triangles are important because they are used in navigation and geodesy (large-area surveying).

Any side of a spherical triangle is less than the sum of the other two sides. **LXIII**

In Fig. 62, the plane angles, AOB, AOC, and BOC, converging at the center of the sphere, O, as a vertex form at that vertex a trihedral angle, $O\text{-}ABC$. Now, in order that we may have a trihedral angle, the property just stated must hold; otherwise, it would be impossible to form a trihedral angle.

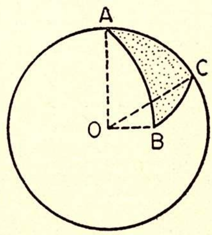

Fig. 61

SHORTEST DISTANCES

We have stated several times that the shortest distance between two points in space is measured along a straight line joining these two points. Now we shall consider the problem of the shortest route between two points that lie on two intersecting planes, such that the entire route (or broken line) lies in the two planes. Such two intersecting planes, M and N, form a dihedral angle, XOY. (Fig. 62a)

In order to determine the shortest distance between the two points, A and B, let us assume that a rubber band is attached to these two points and that it is stretched tightly.

Now flatten out the two faces of the dihedral angle so that they form one plane (Fig. 62b).

Note that the rubber band lies flat in the new plane.

The line of the rubber band, AB, and the line showing the edge of the dihedral angle, OO', intersect at Q and form vertical angles.

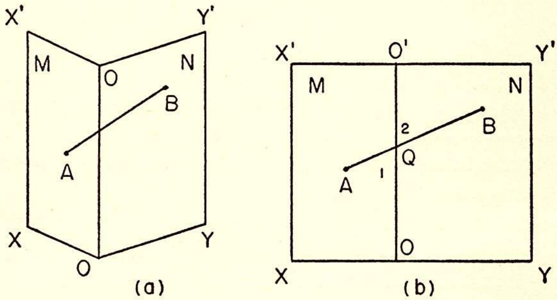

Fig. 62

Thus, we see that the shortest distance between the two points is obtained when the angles, AQO and $O'QB$, as shown in Fig. 62b, are equal. **LXIV**

This same property is obtained if we consider the shortest distance between two points that lie on two non-adjacent faces of a prism. All that we have to do is to flatten out the lateral surface of the prism

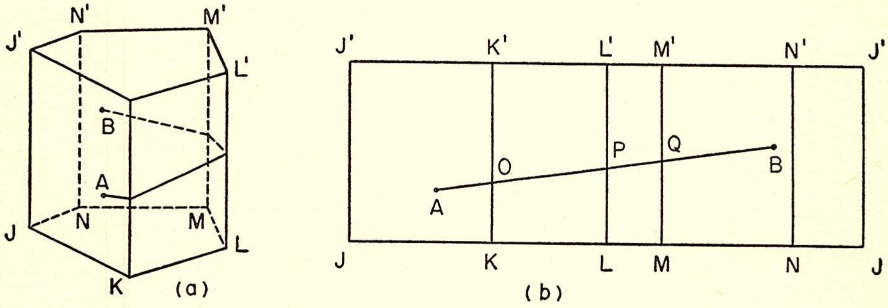

Fig. 63

(Fig. 63). Again, we observe that the equality of the angles (first vertical, and then those formed by parallel lines and transversal) holds.

The shortest distance between two points on the surface of a sphere (measured on the surface of the sphere) is measured along the arc of a great circle joining these two points. **LXII**

· IRRATIONAL AND IMAGINARY NUMBERS ·

By Reginald Stevens Kimball, Ed.D.

SEVERAL times, in earlier issues of PRACTICAL MATHEMATICS, we have approached the subject of roots and powers of numbers. Thus far, we have dealt only with the positive and negative roots of real numbers—integers or fractions. Before we get much closer to the issues which deal with applied mathematics, it will be necessary for us to examine and become familiar with two types of numbers with which we have not yet developed even a passing acquaintance.

IRRATIONAL NUMBERS

Let us take as our starting point a quick review of the Pythagorean theorem, which we met in Issue Number Five (pages 268 and 274). Remembering that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides, we recall also that, for certain number triples, these values always came out with even numbers. The 3–4–5 and the 5–12–13 combinations are the most familiar.

Let us deal with these by way of review:

$$3^2 + 4^2 = 5^2 \qquad\qquad 5^2 + 12^2 = 13^2$$
$$9 + 16 = 25 \qquad\qquad 25 + 144 = 169$$

These relationships hold true if we double the length of each side:

$$6^2 + 8^2 = 10^2 \qquad\qquad 10^2 + 24^2 = 26^2$$
$$36 + 64 = 100 \qquad\qquad 100 + 576 = 676$$

Let us see what happens when we triple each side:

$$9^2 + 12^2 = 15^2 \qquad\qquad 15^2 + 36^2 = 39^2$$
$$81 + 144 = 225 \qquad\qquad 225 + 1296 = 1521$$

We may generalize from these cases that any multiple of the 3–4–5 or the 5–12–13 relationships will still be a Pythagorean number triple. (If you are not convinced, try a few more for yourself!)

The same thing holds true if we use fractions as multiples, or if we divide the lengths of the sides by the same number. Taking a right triangle with sides half as long as our original, we get:

$$(1.5)^2 + 2^2 = (2.5)^2 \qquad\qquad (2.5)^2 + 6^2 = (6.5)^2$$
$$2.25 + 4 = 6.25 \qquad\qquad 6.25 + 36 = 42.25$$

This rule, then, holds good so long as our numbers are multiples or sub-multiples of numbers which have an integral relationship, which can be expressed in either of the given ratios.

Derivation of radical numbers

Although the irrational number does not lend itself to computation in the terms of an exact decimal, it does have a practical bearing

because it helps us to express relationships which exist in actual, practical problems. In this section, we shall consider some of the cases in which we find irrationals. Because these quantities do exist, we shall want to learn how to deal with them. As will be seen from these examples, we find it much simpler to leave the expression as a radicand rather than to compute its decimal value and carry a long, inexact decimal through the whole series of manipulations.

SQUARE ROOT

Let us see what happens when we take some other relationship.

In a 30°-60° right triangle, we know that the hypotenuse is twice the length of the shorter leg. Taking the shorter leg as 1 unit in length, then, we should give the hypotenuse a value of 2. The length of the other leg may then be computed as

$$\sqrt{2^2-1^2}=\sqrt{4-1}=\sqrt{3}.$$

In the case of an isosceles right triangle (a 45°-45° right triangle), we take the unit value for either leg. Then the length of the hypotenuse is

$$\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}.$$

Here we have introduced two symbols which as yet have had no particular meaning to us. It is true that we have been referring to the tables (pages 250 and 251, Issue Number Four) when we have had occasion to use such square-root values, but we may not have realized that they represented something tangible.

Since we cannot express these roots as definite quantities (because their decimal values go on endlessly if we carry them out to an infinite number of places), we call them *irrational* (without exact roots). Just because we cannot measure them exactly by any tool which we yet possess does not mean that they do not exist, as we have just seen when we located their existence as values for a side of a triangle. They are *incommensurable* by any mathematical measuring-stick which we have yet developed.

CUBE ROOT

In the same way, we may wish to show that cube roots may at times be irrational.

Let us examine the cube root of 6 ($\sqrt[3]{6}$) as a specimen.

If $\sqrt[3]{6}$ were rational, we could represent it as the ratio of two integers, such as m and n, representing a fraction reduced to its lowest terms:

$$\sqrt[3]{6}=\frac{m}{n}.$$

Cubing each side of the equation, we should have

$$6=\frac{m^3}{n^3}.$$

Then

$$6n^3 = m^3.$$

Since the left-hand member is multiplied by an even number (6), it must be even no matter whether n is odd or even, and therefore m^3 must be even.

Then the cube root of m^3, or m, must be even.

If m is even, n must be odd; otherwise m and n would both be divisible by 2, and we have already taken $\frac{m}{n}$ as a fraction reduced to its lowest terms.

Since m is even, it is divisible by 2, and we may represent its half by any convenient letter, such as p.

Then $\qquad m = 2p,$

And $\qquad m^3 = 8p^3,$

But $\qquad m^3 = 6n^3,$

$\therefore \qquad 8p^3 = 6n^3,$

Or $\qquad 4p^3 = 3n^3.$

Here we meet an impossible situation: the left-hand member must be even, since it is multiplied by 4; the right-hand member must be odd, since n was odd, the cube of an odd number is odd, and an odd number multiplied by 3 is odd.

Our assumption that the cube root of 6 has an integral ratio is thus proved false, and we know that the cube root of 6 must be irrational. (Note that a cube root of a negative number may be a real negative number.)

We shall be meeting such numbers from time to time as we proceed. We call a number expressed under a radical sign a *surd.*

Fundamental operations with surds

It is sometimes necessary to add, subtract, multiply, or divide surds just as we have already learned to add, subtract, multiply, or divide integers, common fractions, or decimals. If we think of the whole expression under the radical sign as a single term, to be dealt with just as we have dealt with a letter previously (Issue Number Three, pages 135 to 149), we shall have little difficulty in dealing with them.

ADDITION OF RADICALS

When we learned about algebraic addition, we found that we could indicate the addition of unlike letters but could not combine them, that $3a$ added to $4b$ gave us $3a+4b$. Likewise, when we have $\sqrt{2}$ three times and $\sqrt{5}$ four times, we can merely say:

$$3\sqrt{2}+4\sqrt{5}.$$

On the other hand, just as

$$3a+4a=7a,$$

so

$$3\sqrt{2}+4\sqrt{2}=7\sqrt{2}.$$

In performing the following exercises, you will help yourself greatly if you first substitute a letter for each different radical (as above) and

then at the last step substitute the radical for the letter which has been representing it. After some practice of this sort, discard your "crutch" and proceed directly to add the like radicals.

TEST YOUR ABILITY TO ADD RADICALS

1 $2\sqrt{5}+3\sqrt{5}+4\sqrt{5}=?$

2 $6\sqrt{3}+2\sqrt{7}+3\sqrt{3}=?$

3 $\sqrt{5}+2\sqrt{x}+3\sqrt{x}=?$

4 $3\sqrt{y}+3\sqrt{x}+2\sqrt{x}+2\sqrt{y}=?$

5 $5\sqrt{a+b}+3\sqrt{a+b}=?$

6 $3\sqrt{c-d}+2\sqrt{c+d}+7\sqrt{c-d}=?$

SUBTRACTION OF RADICALS

Here, again, we should think of subtraction as negative addition. Then we find that the rules we have just learned about addition apply likewise in subtraction. Just as $4b$ subtracted from $3a$ gives us $3a-4b$, so $4\sqrt{5}$ subtracted from $3\sqrt{2}$ may be merely indicated:

$$3\sqrt{2}-4\sqrt{5},$$

but, just as

$$6a-4a=2a,$$

so

$$6\sqrt{7}-4\sqrt{7}=2\sqrt{7}.$$

TEST YOUR ABILITY TO SUBTRACT RADICALS

7 $9\sqrt{3}-4\sqrt{3}-2\sqrt{3}=?$

8 $8\sqrt{5}-\sqrt{5}-6\sqrt{5}=?$

9 $12\sqrt{x}-3\sqrt{2}-10\sqrt{x}=?$

10 $7\sqrt{x}-4\sqrt{y}-3\sqrt{x}-2\sqrt{y}=?$

11 $6\sqrt{a+b}-3\sqrt{a+b}=?$

12 $5\sqrt{a-b}-2\sqrt{a+b}-3\sqrt{a-b}=?$

MULTIPLICATION OF RADICALS

We may multiply radicals, just like letters, even though they differ in value. However, we are limited by the fact that they must be radicals of the same order (both or all squares, or both or all cubes, or both or all of some other like power) if we are to combine them under the same radical sign. In the case of radicals, we multiply together the coefficients and multiply together the numbers under the radicals. Thus,

$$3\sqrt{2}\cdot 4\sqrt{5}=12\sqrt{10}.$$

If the number under the radical is a perfect square, we proceed to extract the square root, remembering that there are two square roots, represented by the same number, but one positive and one negative.

$$2\sqrt{3}\cdot 3\sqrt{3}=6\sqrt{9}=6\cdot(\pm 3)=\pm 18.$$

$$(3\sqrt{5})(-4\sqrt{5})=-12\sqrt{25}=-12(\pm 5)=\mp 60.$$

In the case of binomials or polynomials, either or both involving radicals, we proceed in just the same way that we did with literal polynomials (pages 142ff of Issue Number Three).

Illustrative Examples

A
$$\begin{array}{l} \sqrt{5}+2\sqrt{3} \\ \sqrt{5}+3\sqrt{3} \\ \hline \sqrt{25}+2\sqrt{15} \\ \qquad 3\sqrt{15}+6\sqrt{9} \\ \hline \sqrt{25}+5\sqrt{15}+6\sqrt{9} = \\ \pm 5+5\sqrt{15}+6(\pm 3) = \\ \pm 5+5\sqrt{15}\pm 18 = \\ \pm 23+5\sqrt{15} \end{array}$$

B
$$\begin{array}{ll} \sqrt{5}+2\sqrt{3} & \\ \sqrt{5}-2\sqrt{3} & \\ \hline \sqrt{25}+2\sqrt{15} & \\ \quad -2\sqrt{15} & -4\sqrt{9} \\ \hline \sqrt{25} & -4\sqrt{9} = \\ \pm 5 & -4(\pm 3) = \\ \pm 5 & \mp 12 = \\ \mp 7 & \end{array}$$

TEST YOUR ABILITY TO MULTIPLY RADICALS

13 $3\sqrt{7}\cdot 2\sqrt{7} = ?$
14 $4\sqrt{3}\cdot 3\sqrt{2} = ?$
15 $2\sqrt{5}\cdot 4\sqrt{7} = ?$
16 $(2\sqrt{3}-3\sqrt{5})(7\sqrt{3}-2\sqrt{5}) = ?$
17 $(3\sqrt{a}+5\sqrt{b})(\sqrt{a}-\sqrt{b}) = ?$
18 $(2\sqrt{7}-3\sqrt{c})(5\sqrt{c}-3\sqrt{7}) = ?$

DIVISION OF RADICALS

Division has been presented several times before as the opposite of multiplication (Issue Number One, page 18; Issue Number Three, page 141). In the case of radicals, this is also true. We may express the division of two radical numbers under one radical sign provided both are radicals of the same order.

$$\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}.$$

TEST YOUR ABILITY TO DIVIDE RADICALS

19 $\sqrt{15} \div \sqrt{5} = ?$
20 $\sqrt{14} \div \sqrt{7} = ?$
21 $\sqrt{3} \div \sqrt{75} = ?$
22 $\sqrt{a} \div \sqrt{b} = ?$
23 $\sqrt{a+b} \div \sqrt{a-b} = ?$
24 $\sqrt[3]{a^4} \div \sqrt[3]{a} = ?$
25 $9\sqrt{6} \div 3\sqrt{2} = ?$
26 $15\sqrt{21} \div 10\sqrt{7} = ?$
27 $\sqrt[5]{b^3} \div \sqrt[5]{b^2} = ?$

Transforming radicals

Sometimes radicals which at first appear unlike may be found to possess certain common factors. Resolving the number under the radical sign to its factors, we find that some of the factors form a perfect square (or higher power), whose root may be removed, leaving a lesser quantity under the radical.

$$3\sqrt{20} = 3\sqrt{2\cdot 2\cdot 5} = 3\sqrt{2^2\cdot 5} = 3\cdot 2\sqrt{5} = 6\sqrt{5}.$$
$$5\sqrt{75}+2\sqrt{27} = 5\sqrt{3\cdot 5\cdot 5}+2\sqrt{3\cdot 3\cdot 3} = 5\cdot 5\sqrt{3}+2\cdot 3\sqrt{3}.$$
$$= 25\sqrt{3}+6\sqrt{3} = 31\sqrt{3}.$$

RATIONALIZING DENOMINATORS

In the case of a fraction under a radical or with a radical in the denominator, we must always make the denominator rational. In changing the denominator to make it rational, we must, obviously, change the numerator also, lest we alter the value of the fraction (Issue Num-

ber One, page 32). To rationalize the denominator, we multiply it by a radical number of the same order sufficient to make the number under the radical sign a perfect square (or cube or higher power, as the case may be). We multiply the numerator by the same amount.

$$\sqrt{\frac{3}{2}}=\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{3\cdot2}}{\sqrt{2\cdot2}}=\frac{\sqrt{6}}{\sqrt{4}}=\frac{\sqrt{6}}{2}\text{ or }\frac{1}{2}\sqrt{6}.$$

$$\frac{3m}{\sqrt{12}}=\frac{3m}{\sqrt{2\cdot2\cdot3}}=\frac{3m}{2\sqrt{3}}=\frac{3m\sqrt{3}}{2\sqrt{3\cdot3}}=\frac{\overset{1}{\cancel{3}}m\sqrt{3}}{2\cdot\cancel{3}_1}=\frac{m\sqrt{3}}{2}.$$

$$\frac{2\sqrt{3}}{\sqrt{5}+1}=\frac{2\sqrt{3}}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1}=\frac{2\sqrt{15}-2\sqrt{3}}{\sqrt{25}-1}=\frac{2\,(\sqrt{15}-\sqrt{3})}{5-1}$$

$$=\frac{\overset{1}{\cancel{2}}(\sqrt{15}-\sqrt{3})}{\cancel{4}_2}=\frac{1}{2}(\sqrt{15}-\sqrt{3}).$$

TEST YOUR KNOWLEDGE OF IRRATIONAL NUMBERS

Transform or rationalize where necessary. Then combine and simplify:

28 $5\sqrt{2}+\sqrt{8}=?$

29 $6\sqrt{5}-4\sqrt{80}=?$

30 $4\sqrt{16a}+3\sqrt{25a}-2\sqrt{9a}=?$

31 $3\sqrt{5a^2}-2\sqrt{9b^2}=?$

32 $\frac{\sqrt{5}}{\sqrt{6}}=?$

33 $\sqrt[3]{\frac{1}{3}}=?$

34 $\frac{2}{\sqrt{3}}=?$

35 $\frac{\sqrt{16}}{\sqrt{9}}=?$

36 $\frac{\sqrt[3]{8}}{\sqrt{4}}=?$

37 $\frac{5}{\sqrt{27}}=?$

38 $\frac{2}{\sqrt{3}-1}=?$

39 $\frac{a+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=?$

40 $\frac{2+\sqrt{5}}{2-\sqrt{3}}=?$

41 $\frac{1}{2}\sqrt{\frac{2}{5}}+\frac{1}{3}\sqrt{\frac{5}{8}}=?$

IMAGINARY NUMBERS

Sometimes we find a negative expression under the radical. Such an expression has, perhaps unfortunately, received the designation of an *imaginary number* because, ordinarily, we do not think of a negative number as lending itself to the process of extraction of a root or roots.

Here again, however, we are dealing with a class of number which we actually meet in practical problems. Remembering our quadratic formula (Issue Number Four, page 199), we know that the expression, $\pm\sqrt{b^2-4ac}$, in the fraction,

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

will be negative whenever $4ac>b^2$.

We may think of this whole expression as equivalent to $\sqrt{-x}$, the quantity found when $4ac$ is subtracted from b^2.

Remembering our definition of square root (Issue Number Two, page 109, and Issue Number Four, page 224), we know that $\sqrt{-x}$ represents a number such that, when it is multiplied by itself, the product will be $-x$.

In ordinary algebra, we know that a negative quantity multiplied by a negative quantity produces a positive result. (Issue Number Three, page 140.) When dealing with radicals, however, in the light of the previous definition, we must remember that

$$\sqrt{-x} \cdot \sqrt{-x} \neq \sqrt{-x \cdot -x},$$

which would give

$$\sqrt{x^2} = \pm x.$$

Rather, we agree that this rule of multiplication is not effective for radicals, and that

$$\sqrt{-x} \cdot \sqrt{-x} = -x.$$

We go a step further, and say that

$$\sqrt{-x} = \sqrt{x \cdot -1} = \sqrt{x} \cdot \sqrt{-1}.$$

Since x may be any number, we see readily that, in the cases considered below, we may resolve the negative radicands to positive radicands multiplied by $\sqrt{-1}$. If the number resulting is a perfect square, we may then extract it from under the radical sign; if it is not a perfect square, we treat it as previously discussed under irrational numbers (pages 358 to 363).

For convenience, to save ourselves the trouble of writing $\sqrt{-1}$ over and over again, we agree that we shall represent $\sqrt{-1}$ by i (standing for *imaginary number*), or, in symbols,

$$\sqrt{-1} = i.$$

Illustrative Example

$\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1} = \sqrt{2} \cdot i = i\sqrt{2}.$

$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1} = \sqrt{3} \cdot i = i\sqrt{3}.$

$\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = \sqrt{4} \cdot i = \pm 2i.$

$\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1} = \sqrt{5} \cdot i = i\sqrt{5}.$

$\sqrt{-6} = \sqrt{6} \cdot \sqrt{-1} = \sqrt{6} \cdot i = i\sqrt{6}.$

$\sqrt{-7} = \sqrt{7} \cdot \sqrt{-1} = \sqrt{7} \cdot i = i\sqrt{7}.$

$\sqrt{-8} = \sqrt{8} \cdot \sqrt{-1} = \sqrt{4 \cdot 2} \cdot i = 2i\sqrt{2}.$

$\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = \sqrt{9} \cdot i = \pm 3i$, etc.

If we have occasion to raise i to a higher power, we get another problem in mathematics.

$$i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1.$$

$$i^3 = i \cdot i \cdot i = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1\sqrt{-1} = -i.$$

$$i^4 = i \cdot i \cdot i \cdot i = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1 \cdot -1 = 1.$$

$$i^5 = i \cdot i \cdot i \cdot i \cdot i = i^4 \cdot i = 1i = \sqrt{-1}.$$

Here we are starting the whole cycle over again. For a convenient rule, we see that we may eliminate i^4 as many times as it is contained in the expression, reducing the quantity to i, i^2, or i^3 and getting its value from the little table above. Thus,

$$i^{13} = i^{12} \cdot i = i = \sqrt{-1}.$$

$$i^{27} = i^{24} \cdot i^3 = i^3 = -i = -\sqrt{-1}.$$

In electrical formulas, we meet I or i representing current. To prevent confusion of i representing current with i representing an imaginary number, we agree in such formulas to replace i with j when discussing imaginary numbers. In such cases, all that we have said here concerning i is applicable to j.

A negative fraction under the radical is dealt with in the same way as any other fraction. Once i has been segregated, we proceed to rationalize the denominator of the resulting radicand.

$$\sqrt{-\frac{2}{3}}=i\sqrt{\frac{2}{3}}=i\sqrt{\frac{2\cdot3}{3\cdot3}}=\frac{i}{3}\sqrt{6}.$$

TEST YOUR ABILITY TO TRANSFORM NEGATIVE RADICALS

Express in the i-form:

42 $\sqrt{-12}=?$
43 $\sqrt{-15}=?$
44 $\sqrt{-27}=?$
45 $\sqrt{-72}=?$
46 $3\sqrt{-8}=?$
47 $5\sqrt{-169}=?$
48 $5\sqrt{-100}=?$
49 $3\sqrt{-60}=?$
50 $\sqrt{-\frac{1}{9}}=?$
51 $\sqrt{-\frac{3}{8}}=?$

Fundamental operations with imaginaries

In adding imaginaries, we resolve them to the i-form, and then treat the i just as we should any other letter in an algebraic expression. This would be a good time to review the work in fundamental operations in algebra (Issue Number Three, pages 135 to 148).

ADDITION OF IMAGINARIES

A $\sqrt{-25}+\sqrt{-49}+\sqrt{-4}=5i+7i+2i=14i.$
B $3\sqrt{-2}+7\sqrt{-5}=3i\sqrt{2}+7i\sqrt{5}.$
C $5\sqrt{-2}+3\sqrt{-8}+3\sqrt{-18}=5i\sqrt{2}+(3\cdot2)i\sqrt{2}+(3\cdot3)i\sqrt{2}$
$=(5i+6i+9i)\sqrt{2}=20i\sqrt{2}.$

SUBTRACTION OF IMAGINARIES

Again, if we think of subtraction as negative addition, we shall have no difficulty in subtracting imaginaries once we have mastered the addition of imaginaries.

A $\sqrt{-25}-\sqrt{-49}-\sqrt{-4}=5i-7i-2i=-4i.$
B $3\sqrt{-2}-7\sqrt{-5}=3i\sqrt{2}-7i\sqrt{5}=i(3\sqrt{2}-7\sqrt{5}).$
C $3\sqrt{-18}-3\sqrt{-8}=(3\cdot3)i\sqrt{2}-(3\cdot2)i\sqrt{2}=(9i-6i)\sqrt{2}=3i\sqrt{2}.$

MULTIPLICATION OF IMAGINARIES

When we are multiplying two numbers, each of which is an imaginary, we have to keep in mind the table of multiples of i which we discussed on page 364, remembering that $i^2=-1$, $i^3=-i$, and $i^4=1$. Aside from this, we proceed as with any problem in multiplication.

A $\sqrt{-9}\cdot\sqrt{-16}=3i\cdot4i=12i^2=12(-1)=-12.$
B $\sqrt{-25}\cdot\sqrt{-49}\cdot\sqrt{-81}=5i\cdot7i\cdot9i=315i^3=315(-i)=-315i.$

DIVISION OF IMAGINARIES

When both dividend and divisor contain i, the i's cancel and we proceed with ordinary integers. If an i remains in the denominator, we must remember that it represents a radical expression and that the denominator must be rationalized. This would be accomplished, as in example D, by multiplying both numerator and denominator by i. Since the i^2 in the denominator equals -1, we write the numerator as a whole number with the sign changed.

A $\dfrac{\sqrt{-100}}{\sqrt{-49}}=\dfrac{10i}{7i}=\dfrac{10}{7}=1\dfrac{3}{7}.$

B $\dfrac{\sqrt{-12}}{\sqrt{-6}}=\dfrac{2\sqrt{-3}}{\sqrt{-6}}=\dfrac{2\sqrt{-3}\cdot\sqrt{-6}}{\sqrt{-6}\cdot\sqrt{-6}}=\dfrac{\overset{1}{\cancel{2}}\sqrt{-18}}{\underset{3}{\cancel{6}}}=\dfrac{\overset{1}{\cancel{3}}\sqrt{-2}}{\cancel{3}}=\sqrt{-2}=2i.$

C $\dfrac{\sqrt{-3}}{\sqrt{-4}}=\dfrac{i\sqrt{3}}{2i}=\dfrac{\sqrt{3}}{2}$ (or) $\dfrac{1}{2}\sqrt{3}.$

D $\dfrac{3}{i}=\dfrac{3\cdot \overset{1}{i}}{i\cdot i}=\dfrac{3i}{-1}=-3i.$

TEST YOUR ABILITY TO COMBINE IMAGINARIES

52 $\sqrt{-4}+\sqrt{-3}=?$

53 $\sqrt{-9}-\sqrt{-4}=?$

54 $\dfrac{\sqrt{-64}}{\sqrt{-25}}=?$

55 $\sqrt{-36}\cdot\sqrt{-9}=?$

56 $\dfrac{i}{\sqrt{-7}}=?$

57 $\dfrac{\sqrt{-20}}{i^2}=?$

58 Solve for x: $5ix=10.$

Complex numbers

When a number is expressed in the form,

$$x+iy,$$

the first part representing a real number and the second part an imaginary, we consider the whole expression a *complex number*. From the treatment that has gone before, we see that any expression, such as

$$x+\sqrt{-y^2},$$

may be written in the above form simply by transforming it.

ADDITION OF COMPLEX NUMBERS

As with any surds, we may add the radical expressions in dealing with imaginaries only in those instances in which the imaginaries may be resolved into comparable numbers. Once we have transformed our expressions to the i-form, we can see immediately the possibilities of combination.

A $(8+\sqrt{-12})+(9+\sqrt{-75})=(8+2\sqrt{-3})+(9+5\sqrt{-3})=17+7i\sqrt{3}.$

B $7+\sqrt{-27}=7+3\sqrt{-3}=7+3i\sqrt{3}.$

C $(4+\sqrt{-25})+(5\sqrt{-16})=4+5i+(5\cdot 4i)=4+25i.$

SUBTRACTION OF COMPLEX NUMBERS

In subtracting complex numbers, we follow the same procedure as that discussed under addition.

A $(10+\sqrt{-4})-(2-3\sqrt{-4})=(10-2i)-(2+6i)$
$=10-2-2i-6i=8-8i=8(1-i).$

B $(5+3\sqrt{-6})-(3+7\sqrt{-24})=(5+3i\sqrt{6})-(3+[7\cdot 2i]\sqrt{6})=2-11i\sqrt{6}.$

MULTIPLICATION OF COMPLEX NUMBERS

By combining what we have just learned about multiplication of imaginaries with what we learned on page 362 about multiplication of surds, we find that the multiplication of complex numbers presents no added difficulties.

A

$$\begin{array}{l} 8-\sqrt{-5} \\ -2+\sqrt{-6} \\ \hline 8-i\sqrt{5} \\ -2+i\sqrt{6} \\ \hline -16+2i\sqrt{5}+8i\sqrt{6}-i^2\sqrt{30} \\ -16+2i\sqrt{5}+8i\sqrt{6}+\sqrt{30} \\ -16+\sqrt{30}+i(2\sqrt{5}+8\sqrt{6}) \end{array}$$

B

$$\begin{array}{l} 4-\sqrt{-6} \\ 3-\sqrt{-27} \\ \hline 4-i\sqrt{6} \\ 3-3i\sqrt{3} \\ \hline 12-3i\sqrt{6}-12i\sqrt{3}+3i^2\sqrt{18} \\ 12-3i\sqrt{6}-12i\sqrt{3}-9\sqrt{2} \\ 12-9\sqrt{2}-i(3\sqrt{6}-12\sqrt{3}) \end{array}$$

DIVISION OF COMPLEX NUMBERS

In dividing by a complex number, we first rationalize the denominator of the expression in its fractional form. We should then recall what we learned in Issue Number Three about the difference of two squares (page 153). If our complex number consists of a real number plus an imaginary, we multiply by the same real number minus the same imaginary. In carrying out the multiplication, we eliminate the radical expression and, since the i^2 equals -1, we change the sign of the coefficient of i^2, which then becomes a real number to be combined with the other real number in the denominator.

A $$\frac{6}{3+\sqrt{-2}}=\frac{6}{3+i\sqrt{2}}=\frac{6(3-i\sqrt{2})}{(3+i\sqrt{2})\,(3-i\sqrt{2})}=\frac{18-6i\sqrt{2}}{9-2i^2}=\frac{18-6i\sqrt{2}}{11}.$$

B $$\frac{4+\sqrt{-3}}{5+\sqrt{-2}}=\frac{(4+i\sqrt{3})\,(5-i\sqrt{2})}{(5+i\sqrt{2})\,(5-i\sqrt{2})}=\frac{20+5i\sqrt{3}-4i\sqrt{2}-i^2\sqrt{6}}{25-2i^2}$$
$$=\frac{20+5i\sqrt{3}-4i\sqrt{2}+\sqrt{6}}{25+2}=\frac{20+i(5\sqrt{3}-4\sqrt{2})+\sqrt{6}}{27}.$$

TEST YOUR KNOWLEDGE OF COMPLEX NUMBERS

59 Add $2+\sqrt{-49}$ and $3+\sqrt{-196}$.
60 Add $3+\sqrt{-36}$ and $4+\sqrt{-81}$.
61 $7-\sqrt{-4}$ minus $3+\sqrt{-4}$.
62 $5+\sqrt{-3}$ minus $2-\sqrt{-5}$.
63 $4+\sqrt{-81}$ times $2+\sqrt{-49}$.
64 $2+2\sqrt{-2}$ times $3+3\sqrt{-3}$.
65 Divide $2+2\sqrt{-2}$ by $3+3\sqrt{-3}$.
66 Divide 9 by $4-\sqrt{-3}$.

The Measuring Rod

1 Desiring to make a sphere, a cabinet-maker selected a cone of wood whose slant height was 6 inches and whose base was 6 inches in diameter. What is the largest sphere which he could make?

2 A graduate is made of a glass cylinder with an inner diameter of 0.65 inches. How far apart should the scratches be made to mark off cubic inches?

3 A marble monument is to be in the shape of a sphere 4 feet in diameter. The sphere is to be cut so as to afford a flat base. If this base must be 1 foot, 6 inches, in diameter, how tall will the monument be?

4 A painter, wishing to estimate the amount of paint required to coat the outside of a conical tank, finds that the depth is 6 feet and the diameter at the top is 5 feet. What is the area to be painted?

5 How many square feet of leather are needed to cover 100 softballs, each 4 inches in diameter, if an allowance of an additional 10% must be made for waste in cutting?

6 How many square feet of an insulating material must be used to cover a cylindrical boiler of height 8 feet and diameter 2 feet, if no allowance for waste is to be made?

7 How many square feet of lumber are needed to build 10 boxes $18'' \times 16'' \times 9''$, allowing for no waste?

8 How many square feet of tin are needed to make a dozen cones, each 6 inches in height and of diameter of 3 inches at the base? Allow for an additional 15% for waste in cutting and seams.

9 A steel waste-basket is 18 inches high and has diameters of 10 inches and 8 inches at the top and bottom. How many square feet of sheet steel are required to make a dozen of these, allowing an additional 10% for the seams?

10 A painter, wishing to estimate the amount of paint needed to cover a hemispherical dome, measures the base and finds its circumference to be 12 feet. For how many square feet of surface should he allow?

11 A barracks is to have a roof sloping at a 30° angle. If the height of the roof at the ridgepole is 16′ and the building is 80′ long, how many square feet of roofing must be used?

12 How many square feet of tin are needed to make 1000 tin cans 6 inches high and 2 inches in diameter if 5 square inches must be allowed for the seams of each can?

13 How many square feet of facing are needed for a store column 2 feet in diameter and 18 feet tall?

14 What volume of concrete is needed for a straight-faced dam, whose top is 40 by 3 feet, whose bottom is 40 by 7 feet and whose height is 16 feet?

15 A bar of copper 4 feet long and 2 square inches in cross-section is to be drawn into wire $\frac{1''}{16}$ in diameter. How many feet of wire will the bar furnish?

16 A swimming tank is 60 feet long, 20 feet wide, 5 feet deep at one end and 12 feet deep at the other. What volume of water is required to fill the tank to within 1 foot of the top?

17 How large a cubical block of steel could be cast from a two-mile span of unused trolley tracks, if the cross-section area of the track is 5 square inches?

18 A cone of brass 8 inches high and 6 inches in diameter at the base is to be recast into a sphere. What will be the surface area of the sphere?

19 A sphere, of specific gravity 0.75, floats in water. What fraction of the radius of the sphere protrudes above the water? (Weight displaced = weight of sphere.)

20 A solid cast-iron cone is drilled along its axis with a hole 2 inches in diameter and 5 inches in depth. If the cross-section of the cone is an equilateral triangle, 1 foot on a side, find the volume of metal removed.

21 A mold when filled with brass to a depth of 8 inches produces a cone weighing 20 pounds. To what depth should it be filled to produce a 10-pound cone?

22 If a brass cone with a height of 6 inches and a radius of 2 inches at the base is cut down to the inscribed hexagonal pyramid, what per cent of the metal is removed?

23 What is the volume of the largest cone of base 3 inches in diameter which can be cut from a cork sphere whose radius is 4 inches?

24 A semi-circle of tin, 4 inches in diameter, is rolled and soldered so as to form a cone. What is the volume of this cone?

25 How many cubic inches of metal are needed for a spherical shell with a thickness of $\frac{1''}{4}$ and an outer diameter of 1 foot?

26 When a hole 1 inch in diameter is drilled through a sphere 4 inches in diameter and weighing 11 pounds, what is the weight of the material removed?

27 If a spherical cheese weighs ten pounds, how far from the center should an army cook cut, in order to slice off three pounds of cheese?

28 Two cylindrical boilers have the same length but the diameter of one is twice the diameter of the other. What are the ratios of their capacities and surface areas?

29 How many spherical buck-shot $\frac{1}{8}$ inch in diameter can be made from a cone of lead 4 inches tall and 3 inches in diameter at the base?

30 A meteorological balloon with radius of 14″ expands to twice its volume on ascending to a high altitude. At this altitude, what is its radius?

31 What is the volume of a barrel whose diameters are 20 inches at the bottom, 25 inches at the middle, and 21 inches at the top, the height being 42 inches?

32 What is the weight of a roller bearing 3 inches long, tapering from a diameter of 1 inch to $\frac{5}{8}$ inch, if the metal used weighs 0.26 lbs. per cubic inch?

33 A length of brass tubing used in the manufacture of shell casings is sufficient for 6 shells. If the inside diameter is 2 inches, the outside diameter is $2\frac{1}{2}$ inches, and the length is 15 inches, what is the weight of brass per shell? (Brass weighs 0.28 lbs. per cubic inch.)

34 How many cubic inches of air are contained in a bell jar 16 inches high and 10 inches in diameter? (The bell jar can be considered a cylinder surmounted by a hemisphere.)

35 The frame of a dump truck has a rectangular lower base, 8 by 12 feet and an upper base 10 by 14 feet. If the depth of the frame is 3 feet, what is the capacity of the truck in cubic yards?

36 A vat in a chemical plant is a wedge with a rectangular top, 2 feet by 3 feet. If the depth is 11 inches, how many gallons of sulphuric acid will the vat hold? (231 cu. in. = 1 gal.)

37 What are the cubic contents of a vat in the shape of a hexagonal pyramid with depth of 4 feet and base 8 inches on side?

38 How much material is removed when 300 round metal disks, each $\frac{1}{2}$ inch thick are trimmed down from 4 inches to $3\frac{1}{2}$ inches in diameter?

39 How many cubic inches of metal are contained in a block shaped as a regular tetrahedron 7 inches on side?

40 How large a regular tetrahedron can be cast from a bar of metal 6 feet long and 8 square inches in cross-section area? Express in terms of the length of a side.

41 A lump of brass in the shape of a cylinder, 4 inches high and 2 inches in radius, is to be melted and cast into the shape of the frustum of a cone, with one base 3 inches in radius and height 5 inches. What will be the radius of the other base?

42 How many gallons of water can be stored in a reservoir whose surface is an ellipse, 18 by 12 feet along the axes, and 6 feet deep? (231 cu. in. = 1 gal.)

43 A bomb crater is 7 feet deep and 11 feet in diameter at the opening. How many cubic yards of earth are needed to fill the crater?

44 The pyramid of Cheops in Egypt was, at the time of erection, 480 feet high and 764 feet in length at the base. What weight of stone was used if the stone weighed 200 lbs. per cubic foot?

45 Every day, a planing machine cuts 3000 grooves 1 foot long, $\frac{1}{2}$ inch wide, and $\frac{1}{4}$ inch deep in a metal part. If the shavings are discarded, how much metal is wasted in one year of 365 days?

46 What is the weight of 100 iron spikes, if the end of each spike is a square $\frac{1}{2}$ inch on a side and the length is 5 inches? (Weight of iron = 0.26 lbs. per cubic inch.)

47 A pile of scrap iron stands 7 feet high and 11 feet in diameter. If we figure that three-quarters of the space occupied is iron, how many cubic feet of iron are in the pile?

48 How many cubic inches of metal are required to cast a flywheel 8 inches in diameter and $\frac{3}{4}$ inch thick supplemented by a flange 1 inch thick and 2 inches broad?

49 If a cubic foot of gold can be hammered into 100,000,000 square inches of gold leaf, what is the thickness of the gold leaf?

50 A workman wishes to replace a cubical metal counterweight with a sphere of the same weight and material. If the cube is 6 inches on a side, what should be the diameter of the sphere?

51 The bed of a stream is approximately an arc of a circle with a radius of 20 feet. If the stream is 20 feet wide at water level and the current is 3 miles per hour, find the amount of water per second carried by the stream.

52 A stone bridge 16 feet long, 10 feet wide, and 9 feet high is supported by 3 arches each 4 feet long and 7 feet high, topped by semi-circular arches. If the stone weighs 200 lbs. per cu. ft., what is the weight of the entire structure?

53 Make an accurate drawing of a cube 6 feet on a side, as seen from a point collinear with a diagonal and 8 feet from the nearest corner.

54 A building is 20 stories tall and has a frontage of 70 feet. There are set-backs of 10 feet and 8 feet, respectively, at the eighth and sixteenth floors. Allowing 10 feet for the height of each story, make an accurate drawing of the building as seen from 80 feet directly in front.

55 What are the two equations whose roots are:

$$2+3i,\ 2-3i;$$

and

$$3+2i,\ 3-2i\ ?$$

56 What are the roots of the following equation:

$$2x^2+5x+7=0?$$

57 Inside a rectangular room in a factory, an electric light is suspended 15 feet above the floor, at a point 10 feet from the south wall of the room and 15 feet from the west wall. The center of a sheet of paper lying on a desk in the same room is 3 feet above the floor, 13 feet from the south wall of the room and 11 feet from the west wall. How far is the electric light from the paper?

58 The length of the boom of a crane from the pivot to the tip is 100 feet. The crane stands at the intersection of two streets with the pivot directly over one corner, and the tip of the boom directly over the corner diagonally opposite. The width of each street is 50 feet. If the pivot of the crane stands 5 feet above the street level, how high is the tip of the crane?

59 An ice cream store uses two different scoops to serve portions. One scoop is in the form of a hemisphere and gives a portion $2\frac{1}{2}$ inches in diameter. The other is in the form of a cone, and the portion is also $2\frac{1}{2}$ inches in diameter at the base of the cone. The volume of each portion is exactly the same. How high is the cone?

60 One corner was cut off a large cube of zinc as a sample, by a plane cut. The corner of the sample was, of course, a corner of the original cube. The three sides of the sample which meet in this corner are each 1 foot long. What is the volume of the sample?

Odd Problems For Off Hours

Optical Illusions

46 How good is your "eye"? Can you detect differences in apparently similar objects readily? In Fig. 64, we have the parallel lines, l_1 and l_2, with arrow points at the ends. Which line is the longer? Try to determine this by inspection before you make the actual measurements.

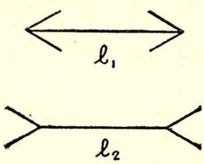

Fig. 64

In Fig. 65, we have a rectangle representing a sheet of cardboard laid over several pipes of different lengths. Can you, without laying a straight edge on the figure, match the two ends of the same pipe? Which pipe does not protrude at both ends?

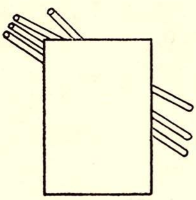
Fig. 65

Can you determine how many cubes are represented in Fig. 66? Taking the black figures as bases, do you get a different count from what you obtain when you take the white figures as bases?

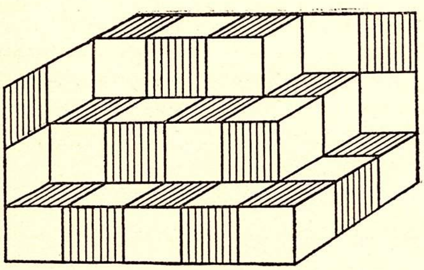
Fig. 66

The two irregularly-shaped figures shown in Fig. 67 should be reproduced in larger cardboard models (See the description of the pantograph on page 375 for fuller instructions) if you are going to try it out on your friends. Using cardboard of different colors will help to intensify the mystery which surrounds the problem. Which of the two figures is the larger? Now measure them!

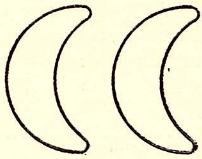
Fig. 67

The Spider and the Fly

47 A prisoner confined in a cell $30' \times 12' \times 12'$ observed a spider on an end wall, equidistant from the two side walls and 1 foot from the ceiling. On the other end wall, 1 foot from the floor, and also equidistant from the side walls, was a fly, stuck in molasses. What is the shortest route which the spider can travel to overtake the fly?

Perfectly Plain

48 Take a perfectly plain square of any size. Cut it into pieces so that you have five smaller squares, all equal, whose total area equals the area of the original square.

Sagging and Rigidity

49 Nail three strips of wood together to form a triangle, as shown in Fig. 68. Notice that the triangle is rigid, withstanding a great amount of pressure.

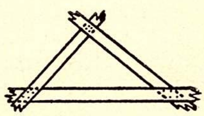
Fig. 68

Now nail four strips together to form a quadrilateral. This "gives" under

pressure if the distances between the fastened points are very great. Remembering that a triangle is rigid, try bracing the quadrilateral by nailing another strip of wood diagonally across the four-sided figure, as shown in Fig. 69, thus securing some of the rigidity of the triangle.

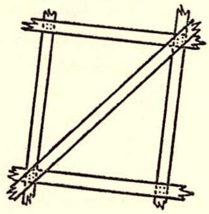

Fig. 69

This is a convenient method in dealing with sagging doors, gates, etc., or for making sure that a packing case will not collapse when weighty objects are placed upon it.

Casing the Rifle

50 A sportsman who desired to ship a prized rifle parcel post was disconcerted to find that he was "up against it". The rifle measured 52 inches in length and parcel-post regulations set a limit of 100 inches combined length and girth for a package weighing under 10 pounds. By calling his knowledge of geometry into play, he was able to package the rifle to conform to the regulations. How did he accomplish it?

Mani-fold Possibilities

51 Take a sheet of typewriter paper (size $8\frac{1}{2}\times 11$). From one corner, measure 4 inches along an end. Mark the point, and turn the remaining $4\frac{1}{2}$ inches of the end up until the point meets the opposite side. Without measuring, compute the length along the diagonal fold. By varying the distance, you may create several interesting problems for speculation.

The solutions to these puzzles will appear in Issue Number Eight

ANSWERS TO PUZZLE-PROBLEMS IN ISSUE NUMBER FIVE

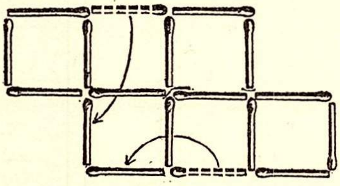

Fig. 70

41 The diagonal line in the rectangle is not a perfectly straight line. (If you cut a cardboard figure sufficiently large, this will be apparent immediately you lay the pieces out.) The "hole" along the diagonal is equivalent to one square unit, thus giving you the extra square.

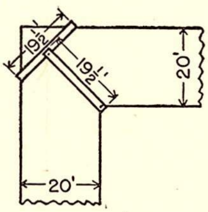

Fig. 71

43 By placing one board across an outside corner of the ditch and laying the second board from the first to the inside corner, the soldiers constructed a safe bridge for their crossing. (See Fig. 71.)

The other problems are explained by the diagrams below.

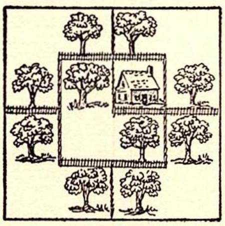

Fig. 72

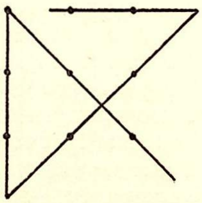

Fig. 73

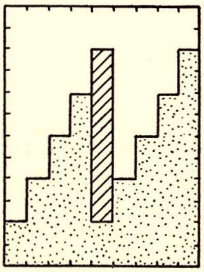

Fig. 74

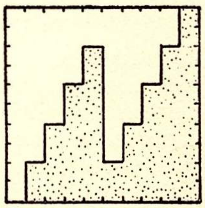

Fig. 75

Strange Ways With Numbers

Straight Lines Without Rulers

26 A mathematician named Peaucellier won the Prix Montyou, awarded by the Institute of France, for the invention of the linkage, a device for drawing straight lines without a ruler. In Fig. 76, the rhombus. $CRPK$, is attached to 2 bars equal in length and meeting at A, so that $AR=AK$. A third vertex, of the rhombus, C, is attached to a point, B on AR so that $BC=AB$. The bars are hinged at all 6 points of inter-

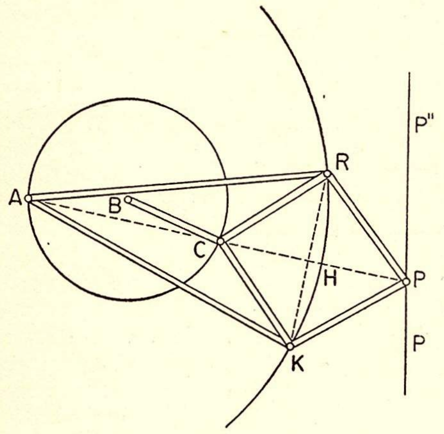

Fig. 76

section. By moving the rhombus, we may use K and R to describe circles around A, and C will move in a circle around B. P will move in a straight line. By attaching a pencil at P, we are thus able to draw a straight line, $P'P''$.

Space Mosaics

27 Cut out of thin cardboard several regular pentagons of exactly the same dimensions. Now try to fit these together so as to enclose a volume of space, the idea being to have each side of each pentagon meet some side of another pentagon, so that the space volume is completely enclosed.

If you place them correctly, you will find that just twelve of these pentagons will fit together to form a complete surface.

Now follow the same procedure with a set of identical regular hexagons. You will not be so fortunate. There is no way to make them fit together into a complete surface.

If you try again with plane figures of seven sides, all "regular", then eight sides, and so on, you will find that none will fit.

The only other regular plane and solid figures out of which complete surfaces can be constructed are triangles and cubes. You are, of course, familiar with the tetrahedron, formed from four triangles, and the cube, formed from six squares. Equilateral triangles can also be made to enclose a complete surface of eight faces; and also one of twenty faces.

Thus, there are in all just five different "regular polyhedra", as listed in Table XXX (page 383). You can use the propositions of solid geometry to prove that only these regular polyhedra are capable of construction.

The Ten-Billionth Power of 2

28 The ancients, who had much less mathematical theory than we to work with, but who had just as much curiosity and ingenuity, knew a great deal about numbers. We have easier ways, and more profitable pastimes, but they had worked out the powers of 2 by successive multiplication away past what anybody now remembers. They even knew how many digits there were in the 10,000,000,000th power of 2. How they determined it is another story, but they had found, correctly, that it contained 3010299956 digits.

Logarithms were invented in the seventeenth century, before the calculus or the infinite series (see Issue Number Four, page 220) with which logarithms can conveniently be calculated. The seventeenth century mathematicians were faced with the problem of calculating logarithms by ordinary arithmetic.

Here is the way they found the logarithm of $2^{10,000,000,000}$. They did not know its mantissa, but its characteristic must be equal to the number of digits in the number. Hence the logarithm of this power of 2 is 3010299956.+, where the + indicates the unknown mantissa. From this they determined that the logarithm of 2, the ten-billionth root of the power, is 0.3010299956, which is correct to ten decimals.

Changing the Scale

29 Known as the pantograph, a simple device for copying figures and at the same time changing the scale

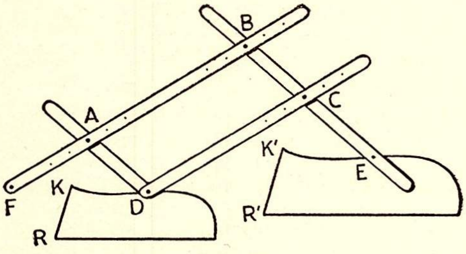

Fig. 77

on which they are drawn makes use of what we have learned about similar figures (page 266). As shown in Fig. 107, the pantograph consists of four bars, joined at the corners by hinges or removable nuts, so that the bars are fastened in the shape of a parallelogram. Some of the commercial models have holes at graduated intervals along the bars, permitting of lengthening or shortening certain sides.

By attaching the pantograph to the drawing board at the fixed point, F, and tracing point D, over the figure to be reproduced, we may insert a pencil at point E.

As we move point D over the outline of the figure, the pencil at E produces an exact copy, which is larger or smaller than the figure being copied, depending upon the scale-points at which we have linked the various bars together.

If you have much occasion to reproduce drawings in various sizes, you will find a pantograph a great help. Commercial models are easily obtainable in most stores selling stationery or drawing supplies.

Proportional Compasses

30 Somewhat similar to the pantograph previously described in the paragraph just above, proportional compasses help in enlarging or reducing reproductions of drawings. As shown in Fig. 77, these compasses are attached one to the other by an adjustable screw, S, which can be moved to any position in the slot. It is evident from what we have already learned in plane geometry that $\triangle$ ASC and DSB are similar, no matter where S be located. That being so, the distance between A and C will always be proportional to the distance between B and D. Using one of the pairs of points to measure a distance, we get a proportionate length between the other pair of points. At what point on AB should S be located to produce a copy of a drawing enlarged 30 per cent, 60 per cent, 150 per cent?

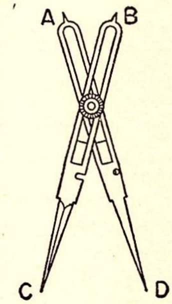

Fig. 78

Solutions to Questions and Exercises in Issue 5

ELEMENTS OF PLANE GEOMETRY

GEOMETRIC TERMS

1 Yes. **5** 180° **8** 15°, 58°
2 −30°, −2°30′ **6** 3, 6 **9** $5\frac{1}{6}°$
3 75°, 150° **7** 0°, 45°, 225° **10** 60°
11 The angle opposite the longest side

PARALLEL LINES

12 Interior angles on same side of transversal
13 Parallels do not meet.
14 720°

CONGRUENCE

15 The three altitudes
The intersection of the altitudes
16 No **18** 2 miles

AREAS

19 1394 sq. in. **22** 2:3
20 60 sq. in. **23** Equal
21 6 sq. in., $1\frac{1}{2}$ sq. in. each. **24** 23:20:3

SIMILAR TRIANGLES

25 5.91 in., 6.61 in.
26 121:625
27 Each line equals $\frac{1}{2}$ the base of the original triangle. $(\frac{1}{2})^2=\frac{1}{4}$
28 Increased $2\frac{1}{4}$:1
29 $\frac{6}{18}=\frac{x}{273}$; $x=91$
30 $\frac{10}{3}=\frac{20}{6}=\frac{50}{h}$ 15 ft.
31 $a^2+b^2=c^2$ 100 ft.
32 $AC=10$, $DC=3.6$;
$\frac{BD}{3.6}=\frac{6.4}{BD}$; $BD=4.8$
34 If $AB \| CD$, the triangles ABO and CDO are similar. We can set up $\frac{AB}{BO}=\frac{CD}{CO}$.

ANGLES IN A CIRCLE

35 45°; yes **37** $\frac{1}{24}$, $\frac{1}{12}$
38 $\sqrt{(4005)^2-(4000)^2}=200+$mi.
39 $\sqrt{(4000+\frac{1}{88})^2-(4000)^2}=9.5$ mi.
40 Central angle = 75°; inscribed = $37\frac{1}{2}°$

PYTHAGOREAN THEOREM

41 (a) $\sqrt{3^2+4^2}=5'$ (b) $\sqrt{5^2+12^2}=13'$
42 $\sqrt{40^2+30^2}=50'$
43 $\sqrt{90^2+90^2}=127.26'$
44 $\sqrt{AC^2-BC^2}=152.7'$
45 $\sqrt{10^2-8^2}=6$
46 $\sqrt{6^2-3^2}=3\sqrt{3}=5.196$
47 $\sqrt{20^2-16^2}=12$
48 $\sqrt{20^2+25^2}=32.02$ ft.
49 No, the sum of the squares of the distances is constant.
50 $2\sqrt{1^2+1^2}=2.828$

FINAL CHECK

56 $180°-57°=123°$
57 Drop a perpendicular from one vertex and find the altitude. Area, $\sqrt{3}=1.732$ sq. mi.
58 $\frac{4}{6}=\frac{x}{5}$; $3\frac{1}{3}''$, $1\frac{2}{3}''$
59 $\sqrt{700^2+300^2}=761.58'$
60 Not unless the angle is the one included between the sides.
61 Let $3x$ be the perimeter.
$\sqrt{x^2-\left(\frac{x}{2}\right)^2}=14.$ 48.496″
62 This is the inscribed circle. Median is $\sqrt{3}$, radius $=\frac{\sqrt{3}}{3}=0.577'$
63 $\pi r^2=8$, $r=1.6$ ft.

THE CONIC SECTIONS

CIRCLES

1 Radius 10 **4** Radius $\frac{\sqrt{5}}{5}=0.447$
2 Radius 13 **5** Radius 0.5
3 Radius 1 **6** Radius $\sqrt{6}$
7 The point (0, 0)
8 Circle with center at (0, 0), radius $\sqrt{26}$
9 Points such as: (2, 3), (−2, −3), (−2, 3), $(\sqrt{13}, 0)$, $(-\sqrt{13}, 0)$, $(0, \sqrt{13})$, $(0, -\sqrt{13})$
10 (4, 2), (−4, −2), (4, −2), $(2\sqrt{5}, 0)$, $(-2\sqrt{5}, 0)$, $(0, 2\sqrt{5})$, $(0, -2\sqrt{5})$
11 (7, 2), (−7, 2), (7, −2), $(\sqrt{53}, 0)$, $(-\sqrt{53}, 0)$, $(0, \sqrt{53})$, $(0, -\sqrt{53})$
12 (−8, −1), (−8, 1), (8, −1), $(\sqrt{65}, 0)$, $(-\sqrt{65}, 0)$, $(0, \sqrt{65})$, $(0, -\sqrt{65})$

OFF-CENTER CIRCLES

13 $(x-2)^2+(y-4)^2=36$
14 $(x+3)^2+(y-5)^2=4$
15 $(x+5)^2+(y+8)^2=1$
16 $(x-6)^2+(y+3)^2=16$
17 $(x-6)^2+y^2=9$
18 $x^2+(y+6)^2=9$
19 Center (1, 3); radius 3
20 Center (1, 2); radius 4
21 Center (−3, 2); radius 5

ELLIPSES

22 Semi-axes: 4, 3
23 Semi-axes: 1, 3
24 Semi-axes: 6, 3
25 Semi-axes: 2, 3
26 Semi-axes: 8, 4
27 Semi-axes: 4, 5
28 Semi-axes: 5, 2
29 Semi-axes: 3, 4
30 $\frac{x^2}{4}+\frac{y^2}{16}=1$
31 $\frac{x^2}{25}+\frac{y^2}{676}=1$
32 $\frac{x^2}{4}+\frac{y^2}{36}=1$
33 $\frac{x^2}{144}+\frac{y^2}{169}=1$
34 No **37** No
35 Yes **38** Yes
36 No **39** No

PARABOLAS

40 Vertex $(-\frac{3}{2}, 0)$, opens right
41 Vertex $(0, 0)$, opens up
42 Vertex $(-\frac{3}{2}, -\frac{1}{4})$ opens up
43 Vertex $(0, -10)$, opens up
44 Vertex $(1, -1)$, opens left
45 Vertex $(-\frac{1}{2}, -2\frac{1}{4})$, opens up
46 Vertex $(0, -1)$, opens up
47 Vertex $(2, 2)$, opens down
48 Vertex $(-6, 0)$, opens left
49 $y=kx^2$; $10=900k$; $y=\frac{x^2}{90}$.
Answers: 38 in., $4\frac{4}{9}$ in.
50 $y=kx^2$; $-225=k(475)^2$;
$y=-\frac{9x^2}{9025}$
51 $-(225\text{-}140)=-\frac{9x^2}{9025}$; $2x=584+$ ft.
52 $ky=x^2$; $8k=100$; $k=12\frac{1}{2}$. $k=4$ times the distance from focus to vertex ($3\frac{1}{8}$ in. from vertex)

HYPERBOLAS

53 1st and 3d quadrants; axes as asymptotes.
54 2d and 4th quadrants; axes as asymptotes.
55 Opens left and right; vertices at $(\pm 4, 0)$; asymptotes: $x=\pm y$.
56 Opens left and right; vertices at $(\pm 3, 0)$; asymptotes: $x=\pm\frac{3}{4}y$.
57 Opens left and right; vertices at $(\pm 4, 0)$; asymptotes: $x=\pm\frac{1}{3}y$.
58 Opens up and down; vertices at $(0, \pm 6)$; asymptotes: $x=\pm\frac{1}{5}y$.
59 Opens up and down; vertices at $(0, \pm 7)$; asymptotes: $x=\pm\frac{1}{7}y$.
60 Circle and tangent line
61 Hyperbola and asymptote
62 Parabola and line
63 Ellipse and line
64 Hyperbola and parabola
65 Circle and hyperbola
66 Parabola and ellipse
67 Circle and ellipse
68 Hyperbola and ellipse

FINAL CHECK

69 $1000\cdot\pi\cdot 5\cdot 12 \div 144 = 1309.0$ sq. ft.
70 Choose (1, 0) as a vertex:
then $\frac{x^2}{1}-\frac{y^2}{b^2}=1$;
substituting: $\frac{9}{1}-\frac{36}{b^2}=1$;
solving: $b^2=4.5$;
$$x^2-\frac{y^2}{4.5}=1$$
71 Choose the plane as the vertex. Let v be the velocity of the plane, and t the time in seconds. Then, at any given moment, $x=vt$, $y=16t^2$.
$$y=kx^2$$
Substituting: $16t^2=kv^2t^2$,
$$k=\frac{16}{v^2}$$
The equation is $y=\frac{16x^2}{v^2}$ where y is the distance the bomb has fallen and x the horizontal distance it has traveled.
72 $y=kx^2$; $7=k\cdot 81$; $\therefore y=\frac{7}{81}x^2$
when $x=4.5$, $y=\frac{7}{4}$.
The lengths should be $(7-\frac{7}{4})=5'3''$.
73 Let x, y be the lengths of the sides of the squares. Then $x^2+y^2=14$. The graph is a circle with center at (0, 0) and radius of $\sqrt{14}$ feet.
74 $C=2\pi\sqrt{\frac{a^2+b^2}{2}}=9.92$ in.
$\frac{9.92}{18}=0.55$ in.

MEASURING ROD

ANGLES

1 The sum of these angles is supplementary to the third angle, and so is the exterior angle. $\therefore 108^\circ+32^\circ=140^\circ$
2 Let $x=$ the base angles. $18+2x=180$; $x=51$. $180^\circ-x=$ the exterior angles $=129^\circ$
3 Draw the figure and note that it is an equilateral triangle. 90 yds.
4 Draw the figure and notice that it is half of an equilateral triangle.
$1000\sqrt{3}=1732$ feet
5 Apply $\frac{180(n-2)}{n}$, where $n=8$; Ans. $=135^\circ$

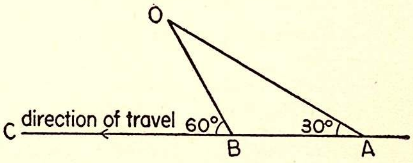

Fig. 78

6 See Fig. 78. $\angle OBA=120^\circ$
$\angle AOB=180^\circ-120^\circ-30^\circ=30^\circ$
Therefore, $\triangle OBA$ is isosceles and $OB=BA$.

TRIANGLES

7 94°, 56°, 30°

8 $10\frac{13}{32}$ sq. in.

9 By construction and measurement: $1\frac{17}{32}$ in.

10 By construction and measurement: $4\frac{1}{32}$ in.

11 By construction: $47\frac{1}{32}$ in.

12 $\frac{74}{360}\cdot 2\pi\cdot 11 = 14.2$ in.

POLYGONS

13 The diagonals of a rhombus are perpendicular. By studying the four triangles they form, we get: 22 sq. ft.

14 Applying the law of Pythagoras, we find that the distance from the plate to second base is 127′3″. Therefore, our answer is: 66′9″.

15 The line cuts off a similar triangle. Since the areas of similar figures are to each other as the squares of their linear dimensions: $\frac{x^2}{21^2}=\frac{A}{9A}$; $x=7$ in.

16 The diagonal is $\sqrt{(6623)^2+(2001)^2}$ (by Pythagoras). The speed is 44 ft. per sec. Applying $t=\frac{D}{r}$, we get 2 min., 37 sec.

17 The distance traveled is the right triangle whose sides are 340 mi., 220 mi., and $\sqrt{(220)^2+(340)^2}$ mi. If the plane had traveled straight out and back, it could have gone to a point 482.5 mi. away.

18 By similar triangles: $\frac{6'}{4'6''}=\frac{x}{28'}$; $x=37'4''$

19 Each time you cut across, you save (140-100) = 40 ft. Your rate of walking is 4.4 ft. per sec. Applying $t=\frac{D}{r}$ and multiplying by 2×365, you get as an answer: 1 hr. 51 min.

20 If the linear dimensions are doubled, the areas of the enlargements will be 4 times as great as the area of the original. $4\times 15\times 24=1440$

21 $\frac{(12000)^2}{(8000)^2}=\frac{324}{x}$; $x=144$

22 The altitude of the triangle is that of a right triangle with hypotenuse 60′ and base 6′, and the correct area is $\frac12\cdot 6\sqrt{(60)^2-6^2}$. According to the Egyptian method, the area is $\frac12(60\times 6)$. The error is only 0.5%.

23 Draw the diagram and construct an altitude at the end of the shorter base. Notice that this altitude is one leg of an isosceles right triangle. $5\frac14$ sq. ft.

24 The areas of all the walls are: $2(9\times 12+9\times 15)=486$. 105% of 486 = 510.3 sq. ft.

25 $\frac{20^2}{1^2}=\frac{x}{30}$. $x=83\frac13$ sq. ft.

CIRCLES

26 Take the centers at (0,0) and (0,2). Then the circles are:

$$x^2+y^2=9$$
$$x^2+(y-2)^2=16$$

Solving these, we find that the circles intersect at $(\pm\frac34\sqrt{15}, -\frac34)$. The distance between these two points is 5.81′.

27 The truck moves the length of one circumference for each revolution of the wheel. Apply $C=2\pi r$, and divide C into 5280′. 560.

28 If the circles are arranged as in Fig. 79, h will be $1.5\sqrt{3}$. Dividing $(54-2r)$ by h, we find that there will be room for 20 rows of circles with 1.6 in. to spare. This 1.6 in. allows us to replace 2 rows

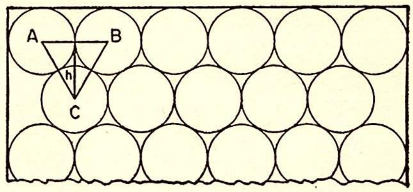

Fig. 79

of 5 by rows of 6, thus getting $12\cdot 6+8\cdot 5=112$ circles. Arranging the circles in vertical and horizontal rows we could cut only $6\cdot 18=108$ circles.

29 The area cut out is one-quarter of the area of a circle of radius 6 in. The area remaining is: $64-\frac{36\pi}{4}=35.73$ sq. in.

30 This circle is the inscribed circle. Draw the diagram and call the radius r. Then the radii to the 3″ side and the 4″ side cut off segments of $(3-r)$ and $(4-r)$, since they form a square with the right angle. Draw a line (from the center) to the angle formed by the 3″ and 5″ sides. Then the two small right triangles formed are congruent. By drawing a similar line to the angle of the 4″ and 5″ sides, we can see that the 5″ side is divided by the radius to it into two segments of length $(3-r)$ and $(4-r)$. Setting this up in the equation: $(3-r)+(4-r)=5$, we see that $r=1$.

31 The distance from this point to the farthest corner may be found by Pythagoras: $\sqrt{9^2+12^2}=15$. Then the area swept over will be $15^2\pi=706.9$ sq. in.

32 Draw the figure. The area can be divided into a right triangle, whose hypotenuse is 26 ft. with one leg being 13 ft. long, and the arc of a circle of radius 26 ft. The triangle is one-half of an equilateral triangle, and therefore the arc of the circle is a 120° arc. The total area equals:
$(\frac{1}{2}\cdot 13\cdot\sqrt{26^2-13^2})+(\frac{120}{360}\cdot\pi\cdot 26^2)=$ 854.26 sq. ft.

33 Let the side of the pentagon be x. Then the area removed is $\frac{108}{360}\pi x^2$, since an angle of a regular pentagon equals $\frac{180\times 3}{5}=108°$. Letting $\frac{108}{360}\pi x^2$ equal 17, we find that $x=4.25$ in.

34 This is the circle inscribed in an equilateral triangle whose altitude is 11 ft. The radius is $\frac{11}{3}$ and the diameter is $7\frac{1}{3}$ ft.

35 Draw the grindstone and a chord of 1 in. Draw radii to the ends of the chords and the center of the arc. Then h is the distance between the chord and arc at the center of the arc. By Pythagoras: $(C)^2+(r-h)^2=r^2$. Substituting our values for C and r, we find that $h^2-20h+\frac{1}{4}=0$. Solving by quadratic formula, $h=0.01251$ in. By the short formula, $h=0.01250$ in. Therefore, the error is only $\frac{1}{100,000}$ of an inch.

PARTS OF CIRCLES

36 Draw two chords, their ends lying within the arc. The perpendicular bisectors of these chords will intersect at same point. Since both bisectors pass through the center, their intersection must be the center, and their lengths from this point to the arc must be the radius.

37 Draw a diagram. Remembering that the tangent is perpendicular to the radius at the point of tangency, drop the perpendicular from the center of the smaller circle to the longer radius. Calculate the areas of the rectangle and triangle thus formed. 30.98 sq. in.

38 Draw the perpendicular bisector of the chord and the radii to the ends of the chord. The radius is the hypotenuse of a right triangle whose side opposite a 60° angle is 8.5 in., or 9.8 in.

39 Draw a diagram, including the line of centers and the radii to the points of contact. We have thus formed two trapezoids with angles of 90°, 90°, 120°, and 60°. Drop a perpendicular from the center of the smaller circle to a radius of the larger. Calculate the length of this perpendicular, which equals the length of the tangent line. The arcs of contact can then be calculated. 126.6″

40 The hexagon is 2 in. on a side. The radii are 2 in. and $\sqrt{3}$ in. Applying $A=\pi r^2$, we get $4\pi-3\pi=\pi$ sq. in.

41 Draw a diagram. The tangents subtend arcs whose sum is 360° and difference 36°. $x+y=360$; $x-y=36$; $x=162°$, $y=198°$. $\therefore$ The length of the arc of contact is: $\frac{198}{360}C=\frac{11}{20}\cdot 10\pi=17.27$ in.

CONICS

42 Draw a diagram. The area of the semicircle less that of the semi-ellipse is $24\pi=75.36$ sq. in.

43 Referring to the equation for an ellipse, take the length of the string as $2a$, which is 7 ft. The distance from a focus to the center of the sheet is $\sqrt{a^2-b^2}=\sqrt{3.5^2-1.5^2}=3.162$ ft.

44 If $x=$ distance from near end, $y=$ height, then $y-21=k(x-294)^2$ is equation of a parabola giving y (height) $=21$ at $x=294$. The parabola also passes through $x=0$, $y=0$; hence $k=-\frac{21}{294^2}=-\frac{1}{4116}$. Equation giving height is $y=21-\frac{(x-294)^2}{4116}$. When $x=72$ feet, $y=9.02$ feet.

45 $y=kx^2$; $10=900k$; $k=\frac{1}{90}$; $y=\frac{x^2}{90}$. Substituting, we get the answers: 38 in., $4\frac{4}{9}$ in.

46 A parabola or a hyperbola.

47 $a=36$, $b=18$. $\frac{x^2}{36^2}+\frac{y^2}{18^2}=1$;
$y^2=18^2\left(1-\frac{x^2}{36^2}\right)$; $y=\sqrt{324-\frac{x^2}{4}}$.

48 Draw a figure. The path is a rectangle with rounded corners. The area can be split up into the original rectangle, four 90° circular arcs, and four rectangular strips. $77+9\pi+42+66=213.27$ sq. ft.

TABLE XXVIII
SLIDE-RULE SETTINGS
ALGEBRA

To Solve:	*A*-Scale	*B*-Scale	*C*-Scale	*CI*-Scale	*D*-Scale	*K*-Scale
$x=a^2$	Read x				under runner set to a.	
$x=a^3$					Under runner set to a,	read x
$x=\sqrt{a}$	Under runner set to a,				read x.	
$x=\sqrt[3]{a}$					Read x	under runner set to a.
$x=\sqrt[3]{a^2}$	Read x					under runner set to a.
$x=\frac{1}{a}$			Read x	under runner set to a.		
$x=\frac{1}{a^2}$		Read x		under runner set to a.		
$x=\frac{1}{\sqrt{a}}$		Under runner set to a,		read x.		
$x=\frac{1}{a^3}$				Under runner set to a,		read x.
$x=\frac{1}{\sqrt[3]{a}}$				Read x		under runner set to a.
$x=ab$			Under b	read x	with a at 1.	
$x=\frac{1}{ab}$			read x	Set a	to b; over 1.	

TABLE XXVIII (Continued)

SLIDE-RULE SETTINGS

To Solve	A-Scale	B-Scale	C-Scale	CI-Scale	D-Scale	K-Scale
$x=\frac{a}{b}$			Set b		to a;	
			under 1		read x.	
$x=\frac{a^2b}{c}$		Set c			to a;	
	read x	over b.				
$x=\frac{\sqrt{ab}}{c}$	Set b	to a;	under c,		read x.	
$a:x=x:b$		Set a			to a;	
$(x=\sqrt{ab})$		under b,			read x.	

GEOMETRY

	A-Scale	B-Scale	C-Scale	D-Scale
Circle			Set 226	over 710;
$C=\pi d$			read diameter;	read circumference.
$A=\pi r^2$	Set 205	to 161;		
		read area		above diameter.
(or)	Set 7854	over 1;	under diameter,	read area.
Inscribed square			Set 99	over 70;
			read diameter;	read side of square.
			Set 40	over 9;
			read circumference;	read side of square.
			Set 322	over 205;
			read area of circle;	read area of square.
Square			Set 70	over 99;
$s\sqrt{2}$			read side of square;	read diagonal of square.

TABLE XXIX

AREAS AND VOLUMES OF SOLID FIGURES

	FACE $A=$	BASE $B=$	LATERAL AREA $S=$	TOTAL AREA $T=$	VOLUME $V=$
Parallelepiped (rectangular)	ah or bh	ab	$2(a+b)h$	$2[(a+b)h+ab]$	abh
Cube	a^2	a^2	$4a^2$	$6a^2$	a^3
Prism	ae	*	pe	$pe+2B$	
Prismatoid					$\frac{h(B_1+4M+B_2)}{6}$
Pyramid	$\frac{bh}{2}$	*	$\frac{nbh}{2}$	$\frac{nbh}{2}+B$	$\frac{Bh}{3}$
Frustum		*	$\frac{(P+p)s}{2}$	$\frac{(P+p)s}{2}+B+b$	$\frac{(B_1+\sqrt{B_1B_2}+B_2)h}{3}$
Cylinder		$2\pi r^2$	$2\pi rh=\pi dh$	$2\pi r(h+r)=\pi d\left(h+\frac{d}{2}\right)$	$\pi r^2h=\frac{\pi d^2h}{4}$
Hollow					$\pi R^2h-\pi r^2h=$ $\pi h(R+r)(R-r)$
Cone	$\pi r\sqrt{r^2+h^2}$	πr^2	$\frac{2\pi rl}{2}=\pi rl$	$\pi r(r+l)$	$\frac{\pi r^2h}{3}$
Frustum		$2\pi r_1^2$ $2\pi r_2^2$	$\pi(r_1+r_2)\sqrt{h^2+(r_1-r_2)^2}$		$\frac{\pi h(r_1^2+r_1r_2+r_2^2)}{3}$ $\frac{\pi h(d_1^2+d_1d_2+d_2^2)}{12}$
			SURFACE $S=$		VOLUME $V=$
Sphere			$4\pi r^2=\pi d^2=12.57r^2$		$\frac{4\pi r^3}{3}=\frac{\pi d^3}{6}=4.189r^3$
Spherical segment			$2\pi rh$		$\frac{\pi h^2(3r-h)}{3}$

*Use Table XXIV, page 316, Issue Number Five.

TABLE XXX
REGULAR POLYHEDRA

	Number of Faces	Nature of Faces	Surface Area $l^2\times$	Volume $l^3\times$
Tetrahedron	4	Equilateral triangles	1.732	0.118
Hexahedron	6	Squares	6.000	1.000
Octahedron	8	Equilateral triangles	3.464	0.471
Dodecahedron	12	Pentagons	20.646	7.663
Icosahedron	20	Equilateral triangles	8.660	2.182

Glossary of Mathematical Terms

area: the number of times which a square on a unit-length of measurement is contained in a given surface. (See pages 336, 382.)

collinear: (of points) lying on the same straight line.

cone: (See pages 318, 343, 348.)

coördinate planes: three planes, mutually perpendicular, on which we locate points in space by indicating their distances from each of the three planes. (See page 326.)

coplanar: lying in the same plane.

cube: a solid object bounded by six planes, having twelve equal edges and all face angles right angles. (See pages 337, 347, 382.)

curved surface: a surface which is not in any part a plane surface.

cylinder: a solid bounded by a cylindrical surface and two parallel intersecting planes; formed by moving a generatrix around a directrix which is a closed curve. (See pages 341, 349, 382.)

cylinder of revolution: a *right circular cylinder* generated by a rectangle that was revolved around one of its sides as an axis. (See page 342.)

cylindrical surface: a surface generated by a straight line which moves parallel to a given straight line (or element) around a given curve. (See page 342.)

dihedral angle: the angle between two planes. If the planes intersect, the angle is measured by the plane angle formed by the lines of intersection of these planes with a plane perpendicular to their line of intersection. If the planes are parallel, the dihedral angle is zero. (See page 330.)

dodecahedron: a polyhedron with twelve faces. In a regular dodecahedron, each face will be a regular pentagon. (See page 383.)

ellipsoid: a surface symmetrical with respect to three mutually perpendicular axes and with the planes determined by them. (See page 355.)

equation of a cylinder and of a sphere: (See page 354.)

face: one of the bounding polygons of a *polyhedron.* (See page 333.)

face angle: the angle between two successive edges of a *polyhedron.* (See page 333.)

frustum: the part of a solid figure which lies between two parallel planes intersecting the solid. (See page 344.)

great circle: a section of a sphere formed by a plane passing through the center of the sphere. (See page 351.)

hexahedron: a polyhedron with six faces. In a regular hexahedron, each face will be a square. (See page 383.)

icosahedron: a polyhedron with twenty faces. In a regular icosahedron, each face will be an equilateral triangle. (See page 383.)

imaginary number: a number representing the square root of a negative number ($\sqrt{-a}$), written ai. (See page 363.)

irrational number: a number which is not *rational;* that is, a number which cannot be expressed as an integer or a terminating decimal. (See page 358.)

lateral area: the area of the *faces* of a parallelepiped or of a *conical* or *cylindrical surface.* (See pages 335, 382.)

lateral edge: the area of the conical or curved surface or of the lateral faces.

lateral face (of a pyramid): any of the bounding triangles (except the base).

lateral surface: the curved surface of a cone or cylinder or the total of the lateral faces of a pyramid or prism.

lune: a portion of a sphere bounded by two great semi-circles.

octahedron: a polyhedron with eight faces. In a regular octahedron, each face will be an equilateral triangle. (See page 383.)

parallelepiped: a *prism* whose *bases* are *parallelograms*; a *polyhedron* whose *faces* are all *parallelograms.* (See pages 339, 348, 382.)

plane angle: the angle formed by drawing a perpendicular in each face of a dihedral angle to the line of intersection of the planes. (See page 330.)

polyhedra: the plural of *polyhedron.*

polyhedral angle: the space bounded by planes converging at a point. (See page 333.)

polyhedron: a solid figure bounded by plane polygons. (See pages 333, 383.)

prism: a polyhedron with two faces (called bases) parallel and congruent, and whose other faces are parallelograms. (See pages 338, 347, 382.)

projection: a point, line, or area in a plane on which a given line, point, or area is projected. (See page 327.)

project (vb.): to depict on one plane a figure which is on another plane.

pyramid: a polyhedron one face of which is a polygon and whose other faces are triangles with a common vertex. (See pages 334, 348, 382.)

radical numbers: numbers under a radical sign. (See *imaginary numbers* and *irrational numbers.)*

rational number: a number that can be expressed as an integer or a common fraction, the quotient of integers. (See page 362.)

regular prism: a right prism with bases which are regular polygons.

right circular cylinder: (See *cylinder of revolution.)*

right prism: a prism with bases perpendicular to the lateral edges. (See page 338.)

slant height (of a right circular cone): the length of an element of the cone. (See page 344.)

sphere: a solid surface which is the locus of points all of which are at the same distance from a fixed point. (See pages 351, 382.)

spherical angle: the figure formed by the intersection of two great circles of a sphere. Measured by the arc of the great circle whose plane is perpendicular to the planes of the intersecting great circles. (See page 356.)

spherical polygon: a portion of a sphere bounded by arcs of great circles. Designations, as triangle, quadrilateral, etc. follow the same definitions as in plane geometry. (See page 351.)

spherical surface: the curved closed surface bounding an area in space. (See page 351.)

spherical wedge: a solid bounded by a lune of the sphere and the planes of two great circles.

spheroid: (See *ellipsoid,* and page 353.)

surd: a number under a radical, indicating a root which can only be approximated. (See page 360.)

surface: plane or curved figures which bound a solid. (See page 321.)

surface area: the sum of the areas of polygons (plane or spherical) which bound a solid. (See pages 337, 353.)

tetrahedral: having four sides. (See page 333.)

tetrahedron: a polyhedron having four faces. (See page 335.)

triagonal: triangular.

triangular prism: a prism having triangular bases.

triangular pyramid: a polyhedron with all four faces triangles.

trihedral: having three sides. (See page 333.)

truncated: a portion of a cone, prism, or pyramid, cut off by non-parallel planes which do not intersect inside the figure. (See page 336.)

volume: the number of times a cube with a side of one unit can be contained in the given solid. (See pages 346ff, 352, 382.)

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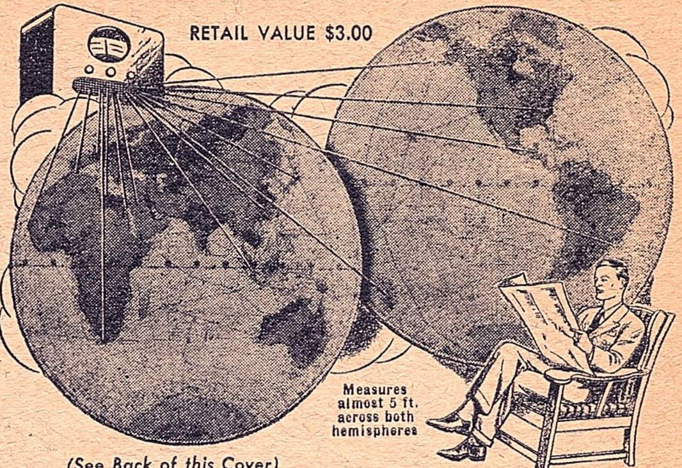

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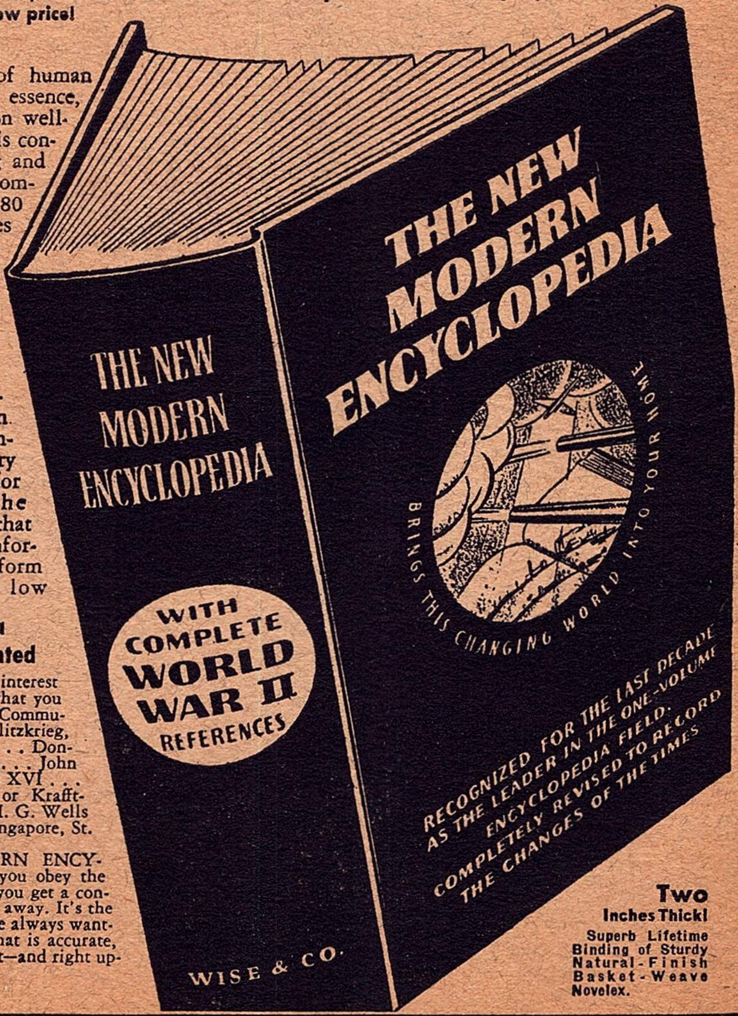

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